GEOMETRIC COMPLEXITY THEORY AND TENSOR RANK

(EXTENDED ABSTRACT)

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Abstract. Mulmuley and Sohoni [28, 29] proposed to view the permanent versus determinant problem as a specific orbit closure problem and to attack it by methods from geometric invariant and representation theory. We adopt these ideas towards the goal of showing lower bounds on the border rank of specific tensors, in particular for matrix multiplication. We thus study specific orbit closure problems for the group $G = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3)$ acting on the tensor product $W = W_1 \otimes W_2 \otimes W_3$ of complex finite dimensional vector spaces. Let $G_s = \mathrm{SL}(W_1) \times \mathrm{SL}(W_2) \times \mathrm{SL}(W_3)$. A key idea from [29] is that the irreducible G_s-representations occurring in the coordinate ring of the G-orbit closure of a stable tensor $w \in W$ are exactly those having a nonzero invariant with respect to the stabilizer group of w.

However, we prove that by considering G_s-representations, as suggested in [28, 29], only trivial lower bounds on border rank can be shown. It is thus necessary to study G-representations, which leads to geometric extension problems that are beyond the scope of the subgroup restriction problems emphasized in [28, 29]. We prove a very modest lower bound on the border rank of matrix multiplication tensors using G-representations. This shows at least that the barrier for G_s-representations can be overcome. To advance, we suggest the coarser approach to replace the semigroup of representations of a tensor by its moment polytope. We prove first results towards determining the moment polytopes of matrix multiplication and unit tensors.

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1. Introduction

Mulmuley and Sohoni [28, 29] proposed to view the permanent versus determinant problem as a specific orbit closure problem and to attack it by methods from geometric invariant and representation theory. So far there has been little progress with this approach, mainly due to the difficulty of the various arising mathematical problems [6]. It is the goal of this paper to examine and to further develop the collection of ideas from [28, 29] at a problem simpler than the permanent versus determinant, but still of considerable interest for complexity theory.

The complexity of matrix multiplication is captured by the rank of the matrix multiplication tensor, a quantity that, despite intense research efforts, is little understood. Strassen [37] already observed that the closely related notion of border rank has a natural formulation as a specific orbit closure problem. Moreover, it is remarkable that the best known lower bound on the rank of matrix multiplication (Bläser [2]) owes its existence to an explicit construction of an invariant polynomial in the vanishing ideal of certain secant varieties (Strassen [36]).

We carried out the program in [28, 29] for the matrix multiplication versus unit tensor problem. More specifically, we determined the stabilizers (symmetry groups) of the corresponding tensors and verified that they are stable. Moreover, we found explicit representation theoretic characterizations of the irreducible G_s-representations occurring in the coordinate rings of the G-orbit closures of these tensors in terms of nonvanishing of Kronecker coefficients and related quantities (G and G_s stand for a product of general linear groups and special linear groups, respectively, cf. (2.1) and §3.5).

Unfortunately, it turns out that using G_s-representations, only trivial lower bounds on border rank can be shown (Theorem 4.6)! This insight is one of our main results. It does not kill the overall program, but implies, unlike proposed in [28, 29], that the finer G-representations have to be considered instead. As a consequence, the stability property is not enough to overcome the issue of orbit closures and additional properties, beyond the subgroup restriction problems emphasized in [28, 29], need to be studied. What we have to face is the problem of extending (highest weight) regular functions from an orbit to its orbit closure. It turns out that this can be captured by a single integer k that seems of a geometric nature (cf. Theorem 7.1). Currently we understand the extension problem very little.

In §6 we prove, for the first time, a lower bound on the border rank of matrix multiplication tensors using G-representations. While this bound is still very modest, it shows at least that the barrier for G_s-representations from Theorem 4.6 can be overcome.

A natural approach to advance is to take a coarser, asymptotic viewpoint which replaces the semigroup of representations by the their moment polytopes [3, 38]. We prove first results towards determining the moment polytopes of matrix multiplication and unit tensors. This is based on the asymptotic properties of the Kronecker polytope derived in [4].

Due to lack of space most of the proofs had to be omitted in this extended abstract. Full proofs are provided in the appendix.

2. Preliminaries

2.1. **Tensor rank.** Let W_1, W_2, W_3 be finite dimensional complex vector spaces of dimensions m_1, m_2, m_3, respectively. We put $W := W_1 \otimes W_2 \otimes W_3$ and call $\underline{m} = (m_1, m_2, m_3)$ the *format* of W. The elements $w \in W$ shall be called *tensors* and w is called indecomposable if it has the form $w = w_1 \otimes w_2 \otimes w_3$. The *rank* $R(w)$ of $w \in W$ is defined as the minimum $r \in \mathbb{N}$ such

that w can be written as a sum of r indecomposable tensors. We note that if $W_3 = \mathbb{C}$, then $R(w)$ is just the rank of the linear map $W_1^* \to W_2$ corresponding to w.

Strassen proved [35] that the minimum number of nonscalar multiplications sufficient to evaluate the bilinear map $W_1^* \times W_2^* \to W_3$ corresponding to w differs from $R(w)$ by at most a factor of two. Determining the rank of specific tensors turns out to be very difficult. Of particular interest are the tensors $\langle n,n,n\rangle \in (\mathbb{C}^{n\times n})^* \otimes (\mathbb{C}^{n\times n})^* \otimes \mathbb{C}^{n\times n}$ describing the multiplication of two n by n matrices. The best known asymptotic upper bound [9] states $R(\langle n,n,n\rangle) = \mathcal{O}(n^{2.38})$, while the best known lower bound [2] is $R(\langle n,n,n\rangle) \geq \frac{5}{2}n^2 - 3n$.

The *border rank* $\underline{R}(w)$ of $w \in W$ is defined as the smallest $r \in \mathbb{N}$ such that w can be obtained as the limit of a sequence $w_k \in W$ with $R(w_k) \leq r$ for all k. Clearly, $\underline{R}(w) \leq R(w)$. Border rank is a natural mathematical notion closely related to the rank and it has played an important role in the discovery of fast algorithms for matrix multiplication, see [5]. We note that the best known lower bound on the border rank of matrix multiplication [25] states that $\underline{R}(\langle n,n,n\rangle) \geq \frac{3}{2}n^2 + \frac{1}{2}n - 1$.

2.2. **Orbit closure problem.** It is possible to rephrase the determination of $\underline{R}(w)$ as an orbit closure problem. Consider the algebraic group

$$G := \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3) \tag{2.1}$$

acting linearly on the vector space $W = W_1 \otimes W_2 \otimes W_3$ via $(g_1,g_2,g_3)(w_1 \otimes w_2 \otimes w_3) := g_1(w_1) \otimes g_2(w_2) \otimes g_3(w_3)$. We shall denote by Gw the *orbit* of w and by $\overline{Gw}$ its *orbit closure*. We say that w is a *degeneration* of v, written $w \trianglelefteq v$, iff $\overline{Gw} \subseteq \overline{Gv}$.

Suppose now $m \leq m_i$ and choose bases $e_1^i, \dots, e_{m_i}^i$ in each of the spaces W_i. The tensor

$$\langle m\rangle := \sum_{j=1}^{r} e_j^1 \otimes e_j^2 \otimes e_j^3, \tag{2.2}$$

is called an *mth unit tensor* in W. Another choice of bases leads to a tensor in the same G-orbit as $\langle m\rangle$, so that the orbit of mth unit tensors in W is a basis independent notion. It is easy to see that $\underline{R}(w) \leq m$ iff $w \trianglelefteq \langle m\rangle$, cf. [37].

3. The GCT program for tensors

We summarize here in a concise way the stepping stones of the GCT program [28, 29], adapted to the tensor setting. For this the review of the GCT program in [6] has been very helpful.

3.1. **Semigroups of representations.** For background on representation theory see [12, 14]. We denote by $V_{\lambda_i}(\mathrm{GL}(W_i))$ the Schur-Weyl module labelled by its highest weight $\lambda_i \in \mathbb{Z}^{m_i}$ (with monotonically decreasing entries). Those yield the rational irreducible G-modules

$$V_{\underline{\lambda}}(G) := V_{\lambda_1}(\mathrm{GL}(W_1)) \otimes V_{\lambda_2}(\mathrm{GL}(W_2)) \otimes V_{\lambda_3}(\mathrm{GL}(W_3)),$$

whose highest weights $\underline{\lambda}$ are triples $\underline{\lambda} = (\lambda_1,\lambda_2,\lambda_3)$. We denote by $V_{\underline{\lambda}}(G)^* = V_{\underline{\lambda}^*}(G)$ the module dual to $V_{\underline{\lambda}}(G)$. Moreover, Λ_G^+ shall denote the semigroup of highest weights of G. For a dimension format $\underline{m}$ we consider the subsemigroup $\Lambda^+(\underline{m}) := \bigcup_{d\in\mathbb{N}} \Lambda_d^+(\underline{m})$ of Λ_G^+, where

$$\Lambda_d^+(\underline{m}) := \big\{\underline{\lambda} = (\lambda_1,\lambda_2,\lambda_3) \mid \lambda_i \vdash_{m_i} d,\ i = 1,2,3\big\}.$$

Here we use the notation $\lambda_i \vdash_{m_i} d$ for a partition $\lambda_i = (\lambda_i^1, \dots, \lambda_i^{m_i})$ of d into at most m_i parts.

The action of G on W induces a linear action of G on the ring $\mathcal{O}(W)$ of polynomial functions on W via $(gf)(w) := f(g^{-1}w)$ for $g \in G$, $f \in \mathcal{O}(W)$, $w \in W$. For any tensor $w \in W$, this defines a linear action of G on the graded ring $\mathcal{O}(\overline{Gw}) = \oplus_{d\in\mathbb{N}}\mathcal{O}(\overline{Gw})_d$ of regular functions on $\overline{Gw}$. (By a regular function on $\overline{Gw}$ we understand a restriction of a polynomial function.) Since G is reductive, the G-module $\mathcal{O}(\overline{Gw})_d$ splits into irreducible G-modules.

We define now the main objects of our investigations.

Definition 3.1. The *semigroup of representations* $S(w)$ of a tensor $w \in W$ is defined as

$$S(w) := \left\{\underline{\lambda} \mid V_{\underline{\lambda}}(G)^* \text{ occurs in } \mathcal{O}(\overline{Gw})\right\}.$$

It is known that $S(w)$ is a finitely generated subsemigroup of $\Lambda^+(\underline{m})$, cf. [3]. It is easy to see that if $V_{\underline{\lambda}}(G)^*$ occurs in degree d, i.e., as a submodule of $\mathcal{O}(\overline{Gw})_d$, then $\underline{\lambda} \in \Lambda_d^+(\underline{m})$.

The general strategy of geometric complexity theory [28] is easily described. Schur's lemma implies that for $w, v \in W$

$$\overline{Gw} \subseteq \overline{Gv} \Longrightarrow S(w) \subseteq S(v). \tag{3.1}$$

In particular, exhibiting some $\underline{\lambda} \in S(w) \setminus S(v)$ proves that $\overline{Gw}$ is not contained in $\overline{Gv}$. If $v = \langle m\rangle$, this establishes the lower bound $\underline{R}(w) > m$. We call such $\underline{\lambda}$ a *representation theoretic obstruction*. We note that a more refined approach would be to study the multiplicity of $V_{\underline{\lambda}}(G)^*$ in $\mathcal{O}(\overline{Gw})$, which can only decrease under degenerations.

3.2. **Kronecker semigroup.** Let $[\lambda_i]$ denote the irreducible representation of the symmetric group S_d on d letters labelled by the partition $\lambda_i \vdash d$. For $\underline{\lambda} \in \Lambda_d^+(\underline{m})$ we define the *Kronecker coefficient* $g(\underline{\lambda})$ as the dimension of the space of S_d-invariants of the tensor product $[\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3]$. It is a well known fact that $g(\underline{\lambda}) = \mathrm{mult}(V_{\underline{\lambda}}(G)^*, \mathcal{O}(W)_d)$, see [24] or (10.3). The *Kronecker semigroup* of format $\underline{m}$ is defined by $K(\underline{m}) := \bigcup_{d\in\mathbb{N}}\left\{\underline{\lambda} \in \Lambda_d^+(\underline{m}) \mid g(\underline{\lambda}) \neq 0\right\}$.

Lemma 3.2. *We have $S(w) \subseteq K(\underline{m})$ with equality holding for Zariski almost all $w \in W$.*

3.3. **Inheritance.** For applying the criterion "$\underline{R}(w) \le r$ iff $w \trianglelefteq \langle r\rangle$" from §2.2, we need to understand how $S(w)$ changes when we embed $w \in W$ in a larger space. Fortunately, when properly interpreted, nothing happens. Suppose that W_i is a subspace of W_i', put $m_i' := \dim W_i'$, and let

$$W' := W_1' \otimes W_2' \otimes W_3', \quad G' := \mathrm{GL}(W_1') \times \mathrm{GL}(W_2') \times \mathrm{GL}(W_3').$$

Let w' denote the image of $w \in W$ under the embedding $W \hookrightarrow W'$. A highest G-weight $\underline{\lambda}$ with nonnegative entries can be interpreted as a highest G'-weight $\underline{\lambda}$ by appending zeros to the partitions λ_i. We may thus interpret $\Lambda^+(\underline{m})$ as a subset of $\Lambda^+(\underline{m'})$.

Proposition 3.3. *With the above conventions we have $S(w) = S(w')$.*

This result can be shown similarly as in [29, 24, 6].

3.4. **Stabilizer and invariants.** As a first approach towards understanding $S(w)$ we may replace the orbit closure $\overline{Gw}$ by the orbit Gw and focus on the representations occuring in the ring $\mathcal{O}(Gw)$ of regular functions on Gw. (A regular function on Gw is a function that is locally rational, cf. [15, p. 15].) This leads to definition of the *auxiliary semigroup of representations:*

$$S^o(w) := \big\{\underline{\lambda} \in \Lambda_G^+ \mid V_{\underline{\lambda}}(G)^* \text{ occurs in } \mathcal{O}(Gw)\big\}.$$

$S^o(w)$ is finitely generated [3] and clearly contains $S(w)$.

The *stabilizer* of w is defined as $H := \mathrm{stab}(w) := \{g \in G \mid gw = w\}$. Let $V_{\underline{\lambda}}(G)^H$ denote the space of H-invariants in $V_{\underline{\lambda}}(G)$. The next characterization follows from the algebraic Peter-Weyl Theorem for G as in [6].

Proposition 3.4. *We have* $S^o(w) = \big\{\underline{\lambda} \in \Lambda_G^+ \mid V_{\underline{\lambda}}(G)^H \neq 0\big\}$.

3.5. **Stability.** Consider the subgroup $G_s := \mathrm{SL}(W_1) \times \mathrm{SL}(W_2) \times \mathrm{SL}(W_3)$ of G.

Definition 3.5. A tensor $w \in W$ is called *stable*, iff $G_s w$ is closed.

Consider the residue class map $\prod_{i=1}^3 \mathbb{Z}^{m_i} \to \prod_{i=1}^3 \mathbb{Z}^{m_i}/\mathbb{Z}\varepsilon_{m_i}$, where $\varepsilon_{m_i} := (1,\dots,1)$. When interpreting highest weights of G_s-modules appropriately, this defines a surjective morphism $\pi\colon \Lambda_G^+ \to \Lambda_{G_s}^+$ of the semigroup of highest weights of G and G_s, respectively.

We put $S_s(w) := \pi(S(w))$ and $S_s^o(w) := \pi(S^o(w))$. These semigroups describe the irreducible G_s-modules occurring in $\mathcal{O}(\overline{Gw})$ and $\mathcal{O}(Gw)$, respectively. However, when going over to G_s-modules, the information about the degree d in which the modules occur is lost.

Proposition 3.6. *If w is stable, then $S_s(w) = S_s^o(w)$.*

Proof. Put $\varepsilon_{\underline{m}} := (\varepsilon_{m_1}, \varepsilon_{m_2}, \varepsilon_{m_3})$. The assertion is equivalent to the statement

$$\forall \underline{\lambda} \in S^o(w)\ \exists k \in \mathbb{Z} \quad \underline{\lambda} + k\varepsilon_{\underline{m}} \in S(w). \tag{3.2}$$

Suppose that $\underline{\lambda} \in S^o(w)$. Then $V_{\underline{\lambda}}(G)^*$ occurs in $\mathcal{O}(Gw)_d$ for some $d \in \mathbb{Z}$. Let $f \in \mathcal{O}(Gw)_d$ be a highest weight vector of $V_{\underline{\lambda}}(G)^*$. The restriction $\tilde f$ of f to $G_s w$ does not vanish since Gw is the cone generated by $G_s w$. So $\tilde f$ is a highest weight vector and $V_{\pi(\underline{\lambda})}(G_s)^*$ occurs in $\mathcal{O}(G_s w)$.

The G_s-equivariant restriction morphism $\mathcal{O}(\overline{Gw}) \to \mathcal{O}(G_s w)$ is surjective since $G_s w$ is assumed to be closed. It follows that $\mathcal{O}(\overline{Gw})$ contains an irreducible module $V_{\pi(\underline{\lambda})}(G_s)^*$. This means that $V_{\underline{\lambda}+k\varepsilon_{\underline{m}}}(G)^*$ occurs in $\mathcal{O}(\overline{Gw})$ for some $k \in \mathbb{Z}$. $\square$

Combining this with Proposition 3.4, we obtain a characterization of $S_s(w)$ for stable tensors w, which only involves the stabilizer H of w. The problem is reduced to the question of which $V_{\underline{\lambda}}(G)$ contain nonzero H-invariants.

4. Unit tensors

4.1. **Stabilizer and stability.** Suppose $W_i = \mathbb{C}^m$, $G_m := \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_m$, and recall the definition of the mth unit tensor $\langle m\rangle$ from (2.2). Let $P_\pi \in \mathrm{GL}_m$ denote the permutation matrix corresponding to $\pi \in S_m$.

Proposition 4.1. *The stabilizer H_m of $\langle m\rangle$ is the semidirect product of the normal divisor*

$$D_m := \{(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \mid \forall i\ a_i b_i c_i = 1\},$$

and the symmetric group S_m diagonally embedded in G_m via $\pi \mapsto (P_\pi, P_\pi, P_\pi)$.

We remark that $\langle m\rangle$ is uniquely determined by its stabilizer up to a scalar.

Proposition 4.2. *The unit tensor $\langle m\rangle$ is stable.*

This follows from Kempf's refinement [19] of the Hilbert-Mumford criterion.

4.2. **Representations.** Let $\mathsf{Par}_m(d)$ denote the set of partitions of d into at most m parts. The *dominance order* $\preceq$ on $\mathsf{Par}_m(d)$ is defined by $\lambda \preceq \mu$ iff $\sum_{j=1}^k \lambda^j \le \sum_{j=1}^k \mu^j$ for all k. This defines a lattice, in particular two partitions λ, μ have a well defined meet $\lambda \curlywedge \mu$, cf. [34]. We call $\alpha \in \mathsf{Par}_m(d)$ *regular* if its components are pairwise distinct.

Lemma 4.3. *(1) The set of regular partitions in $\mathsf{Par}_m(d)$ has a unique smallest element $\perp_m(d)$.*
(2) For any $\lambda \in \mathsf{Par}_m(d)$ we have $\perp_{m+1}(d+km) \preceq (\lambda^1+k,\dots,\lambda^m+k,0)$ for sufficiently large k.

Let T_m denote the maximal torus of GL_m of diagonal matrices. For $\alpha \in \mathbb{Z}^m$ with $|\alpha| := \sum_j \alpha_j = d$ and $\lambda \in \mathsf{Par}_m(d)$ one defines the *weight space* of $V_\lambda = V_\lambda(\mathrm{GL}_m)$ for the weight α as

$$V_\lambda^\alpha := \big\{ v \in V_\lambda \mid \forall t \in T_m \ t\cdot v = t^\alpha \big\}.$$

Here we used the shorthand notation $t = \mathrm{diag}(t_1,\dots,t_m)$ and $t^\alpha = t_1^{\alpha_1}\cdots t_m^{\alpha_m}$. It is well known that V_λ decomposes as $V_\lambda = \bigoplus_\alpha V_\lambda^\alpha$. Moreover, V_λ^α is nonzero iff $\alpha \preceq \lambda$, cf. [12].

The symmetric group S_m acts on $\mathbb{Z}^m$ by permutation, namely $(\pi\alpha)(i) := \alpha(\pi^{-1}(i))$ for $\pi \in S_m$. It is easy to check that $P_\pi V_\lambda^\alpha = V_\lambda^{\pi\alpha}$. In particular, the stabilizer $\mathrm{stab}(\alpha)$ of α leaves V_λ^α invariant. Note that $\mathrm{stab}(\alpha)$ is trivial iff α is regular.

Theorem 4.4. *For $\underline{\lambda} \in \Lambda_d(m,m,m)$ we have $\dim(V_{\underline{\lambda}})^{H_m} = \sum_\alpha \dim\big(V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha\big)^{\mathrm{stab}(\alpha)}$, where the sum is over all $\alpha \in \mathsf{Par}_m(d)$ such that $\alpha \preceq \lambda_1 \curlywedge \lambda_2 \curlywedge \lambda_3$.*

The next result shows that the highest weights outside the auxiliary semigroup of the unit tensors are very rare.

Corollary 4.5. *(1) If there is a regular $\alpha \preceq \lambda_1 \curlywedge \lambda_2 \curlywedge \lambda_3$, then $\underline{\lambda} \in S^o(\langle m\rangle)$.*
(2) If $\lambda_1,\lambda_2,\lambda_3$ are all regular, then $\underline{\lambda} \in S^o(\langle m\rangle)$.

Proof. (1) is an immediate consequence of Proposition 3.4 and Theorem 4.4. If $\lambda_1,\lambda_2,\lambda_3$ are all regular, then $\perp_m(d) \preceq \lambda_i$ for $i=1,2,3$ by Lemma 4.3(1). Now apply (1). ☐

Theorem 4.6. *For any $\underline{\lambda} \in \Lambda_d^+(m,m,m)$ there exists $k \in \mathbb{N}$ such that $\underline{\lambda}' \in S^o(\langle m+1\rangle)$, where $\lambda_i = (\lambda_i^1,\dots,\lambda_i^m)$ and $\lambda_i' = (\lambda_i^1+k,\dots,\lambda_i^m+k,0)$.*

Proof. Lemma 4.3(2) implies $\perp_{m+1}(d+km) \preceq \lambda_1',\lambda_2',\lambda_3'$ for sufficiently large k. Now apply Corollary 4.5(1). ☐

Theorem 4.6 has severe consequences. It tells us that for any tensor w of format (m,m,m), the trivial lower bound $\underline{R}(w) > m$ is the best that can be shown using G_s-obstructions!

5. Matrix multiplication tensors

We fix complex vector spaces U_i of dimension n_i, put $W_{12} := U_1^* \otimes U_2,\ W_{23} := U_2^* \otimes U_3,\ W_{31} := U_3^* \otimes U_1$, and consider the group

$$G := \mathrm{GL}(U_1^* \otimes U_2) \times \mathrm{GL}(U_2^* \otimes U_3) \times \mathrm{GL}(U_3^* \otimes U_1).$$

acting on $W := W_{12} \otimes W_{23} \otimes W_{31}$. We define the *matrix multiplication tensor* $\mathrm{M}_{\underline{U}} \in W := W_{12} \otimes W_{23} \otimes W_{31}$ as the tensor corresponding to the linear form

$$W^* \to \mathbb{C},\ \ell_1 \otimes u_2 \otimes \ell_2 \otimes u_3 \otimes \ell_3 \otimes u_1 \mapsto \ell_1(u_1)\ell_2(u_2)\ell_3(u_3),$$

obtained as the product of three contractions ($\ell_i \in U_i^*$ and $u_i \in U_i$). To justify the naming we note that, using the canonical isomorphisms $\mathrm{Hom}(U_2, U_1) \simeq U_1 \otimes U_2^*$ and $\mathrm{Bil}(U, V; W) \simeq U^* \otimes V^* \otimes W$, one easily checks that $\mathrm{M}_{\underline{U}}$ corresponds to the bilinear map

$$\mathrm{M}_{\underline{U}} \colon \mathrm{Hom}(U_2, U_1) \times \mathrm{Hom}(U_3, U_2) \to \mathrm{Hom}(U_3, U_1),\ (\varphi, \psi) \mapsto \varphi \circ \psi$$

describing the composition of linear maps (note that we exchanged the order for the third factor: $\mathrm{Hom}(U_3, U_1) \simeq U_3^* \otimes U_1$). If $U_i = \mathbb{C}^{n_i}$, then this bilinear map corresponds to the multiplication of $n_1 \times n_2$ with $n_2 \times n_3$ matrices. In this case we shall write $\langle n_1, n_2, n_3 \rangle = \mathrm{M}_{\underline{U}}$.

5.1. **Stabilizer and stability.** We put $K := \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ and consider the following morphism of groups

$$\Phi \colon K \to G,\ (a_1, a_2, a_3) \mapsto ((a_1^*)^{-1} \otimes a_2, (a_2^*)^{-1} \otimes a_3, (a_3^*)^{-1} \otimes a_1) \tag{5.1}$$

with the kernel $\mathbb{C}^\times \cdot \mathrm{id} \simeq \mathbb{C}^\times$. Note that $\mathrm{GL}(U_i)$ acts on $U_i^* \otimes U_i$ in the following way:

$$a_i \cdot (\ell_i \otimes u_i) = \big((a_i^{-1})^* \otimes a_i\big)(\ell_i \otimes u_i) = (a_i^{-1})^*(\ell_i) \otimes a_i(u) = (\ell_i \circ a_i^{-1}) \otimes a_i(u_i).$$

Hence this action leaves the trace $U_i^* \otimes U_i \to \mathbb{C}, \ell_i \otimes u_i \mapsto \ell_i(u_i)$ invariant. This implies that the image of Φ is contained in the stabilizer H of $\mathrm{M}_{\underline{U}}$. In fact, equality holds.

Proposition 5.1. *The stabilizer H of $\mathrm{M}_{\underline{U}}$ equals the image of Φ. In particular, $H \simeq K/\mathbb{C}^\times$.*

In the cubic case, this is a consequence of [10]. We remark that $\mathrm{M}_{\underline{U}}$ is uniquely determined by its stabilizer up to a scalar.

Proposition 5.2. *The matrix multiplication tensor $\mathrm{M}_{\underline{U}}$ is stable.*

This can be shown by Kempf's refinement [19] of the Hilbert-Mumford criterion, cf. [27].

5.2. **Representations.** Suppose that $\lambda_{12} \in \mathbb{Z}^{n_1 n_2}$ is a highest weight vector for $\mathrm{GL}(U_1^* \otimes U_2)$ and $\lambda_{23} \in \mathbb{Z}^{n_2 n_3}$, $\lambda_{31} \in \mathbb{Z}^{n_3 n_1}$ are highest weight vectors for $\mathrm{GL}(U_2^* \otimes U_3)$ and $\mathrm{GL}(U_3^* \otimes U_1)$, respectively. Put $\underline{\lambda} = (\lambda_{12}, \lambda_{23}, \lambda_{31})$ and consider the irreducible G-module

$$V_{\underline{\lambda}} := V_{\lambda_{12}}(\mathrm{GL}(U_1^* \otimes U_2)) \otimes V_{\lambda_{23}}(\mathrm{GL}(U_2^* \otimes U_3)) \otimes V_{\lambda_{31}}(\mathrm{GL}(U_3^* \otimes U_1)).$$

Theorem 5.3. *Let $\lambda_{12}, \lambda_{23}, \lambda_{31}$ be partitions of d and H be the stabilizer of $\mathrm{M}_{\underline{U}}$. Then*

$$\dim(V_{\underline{\lambda}})^H = \sum_{\mu_1 \vdash_{n_1} d, \mu_2 \vdash_{n_2} d, \mu_3 \vdash_{n_3} d} g(\lambda_{12}, \mu_1, \mu_2) \cdot g(\lambda_{23}, \mu_2, \mu_3) \cdot g(\lambda_{31}, \mu_3, \mu_1).$$

6. A few examples and computations

6.1. **A family of G-obstructions.** We use the frequency notation $k_1^{e_1} k_2^{e_2} \cdots k_s^{e_s}$ to denote the partition of $\sum_i k_i e_i$ where k_i occurs e_i times.

Lemma 6.1. *We have $\underline{\lambda}_n := (2^{n^2}0, 2^{n^2}0, (2n^2-3)^1 1^3 0^{n^2-3}) \in S(\langle n, n, n \rangle) \setminus S^o(\langle n^2 + 1 \rangle)$ for $n \geq 2$. This implies $\underline{R}(\langle n, n, n \rangle) > n^2 + 1$.*

Proof. (Sketch) 1. We apply Theorem 4.4. Put $N := n^2$ and $\underline{\lambda}_n = (\lambda_1, \lambda_2, \lambda_3)$. The only partitions $\alpha \in \mathsf{Par}_{N+1}(2N)$ smaller than $\lambda_1, \lambda_2, \lambda_3$ are $2^N 0$ and $2^{N-1}1^2$. A computation using the tableaux straightening algorithm from [11, p.110] shows that $(V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha)^{\mathrm{stab}(\alpha)} = 0$ for both α. Proposition 3.4 tells us that $\underline{\lambda}_n \notin S^o(\langle n^2+1\rangle)$.

2. Using [31, 33] one can show $g(\underline{\lambda}_n) = 1$. Hence the highest weight vector $f \in \mathcal{O}(W)$ of weight $\underline{\lambda}_n$ is uniquely determined up to a scalar. We explicitly constructed f and (guided by computer calculations) proved that $f(\langle n,n,n\rangle) \neq 0$. Hence $\underline{\lambda}_n \in S(\langle n,n,n\rangle)$. For the lower bound on $\underline{R}$ apply (3.1). □

Remark 6.2. 1. Theorem 5.3 with $\mu_i = (2n)^n$, a positivity proof for the resulting Kronecker coefficients, and Proposition 3.4 yield $\underline{\lambda}_n \in S^o(\langle n,n,n\rangle)$. In order to guarantee $\underline{\lambda}_n \in S(\langle n,n,n\rangle)$ we currently know of no better way than to evaluate a highest weight vector at $\langle n,n,n\rangle$. In general, this becomes prohibitively costly for larger dimension formats.

2. Lemma 6.1 yields $\underline{R}(\langle 2,2,2\rangle) > 5$. It is known [23] that $\underline{R}(\langle 2,2,2\rangle) = 7$. So far we have been unable to reach the optimal lower bound by an obstruction.

6.2. **Strassen's invariant.** Let $W = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^3$, $m \geq 3$, and consider $\underline{\lambda}_m := (3^m, 3^m, m^3)$. Strassen [36] constructed an explicit invariant $f_m \in \mathcal{O}(W)$ of highest weight $\underline{\lambda}_m^*$, that vanishes on all tensors in W with border rank at most $r = \lceil 3m/2 \rceil - 1$. Hence $f_m(w) \neq 0$ implies $\underline{R}(w) > r$.

Let $\underline{\lambda}'_m \in \Lambda^+(r,r,r)$ be obtained from $\underline{\lambda}_m$ by appending zeros. It is tempting to conjecture that $\underline{\lambda}'_m \notin S(\langle r\rangle)$, because then Strassen's implication would be a consequence of the existence of the obstruction $\underline{\lambda}_m$. Indeed, $f_m(w) \neq 0$ implies $\underline{\lambda}_m \in S(w)$ and, assuming the conjecture, $\underline{\lambda}_m \in S(w) \setminus S(\langle r\rangle)$ and thus $\underline{R}(w) > r$. Unfortunately, the conjecture is already false for $m = 4$! An extensive computer calculation revealed the existence of $\tilde{f}_4 \in \mathcal{O}(W)_{12}$ of highest weight $\underline{\lambda}_4^*$ and $g \in G$ such that $\tilde{f}_4(g\langle 5\rangle) \neq 0$, which shows $\underline{\lambda}'_4 \in S(\langle 5\rangle)$. Note $g(\underline{\lambda}_4) = 2$.

7. Extension problem and nonnormality

In order to advance, we need to study the difference between $S(w)$ and $S^o(w)$. Let W be of format $\underline{m}$ and $w \in W$ be stable. If $\underline{\lambda} \in S^o(w)$, then Proposition 3.6 implies that there exists $k \in \mathbb{Z}$ such that $\underline{\lambda} + k\varepsilon_{\underline{m}} \in S(w)$, where $\varepsilon_{\underline{m}} = (\varepsilon_{m_1}, \varepsilon_{m_2}, \varepsilon_{m_3})$. It is of interest to know the smallest such k. Below we will see that k can be given a geometric interpretation in terms of the problem of extending regular functions from Gw to $\overline{Gw}$.

We call the group morphism $\det\colon G \to \mathbb{C}^\times, (g_1, g_2, g_3) \mapsto \det g_1 \det g_2 \det g_3$ the determinant on G. In the following we will assume that $\varepsilon_{\underline{m}} \in S^o(w)$. By Proposition 3.4 this is equivalent to $\det g = 1$ for all $g \in \mathrm{stab}(w)$. We note that this condition is satisfied for $w = \langle n,n,n\rangle$ due to Proposition 5.1.

If $\varepsilon_{\underline{m}} \in S^o(w)$, then det induces the well defined regular function $\det_w\colon Gw \to \mathbb{C}, gw \mapsto \det g$.

Theorem 7.1. *Suppose that $w \in W$ is a stable tensor and $\varepsilon_{\underline{m}} \in S^o(w)$.*

(1) *Then w has the cubic format (m,m,m).*
(2) *The extension of $\det_w$ to $\overline{Gw}$ with value 0 on the boundary $\overline{Gw} \setminus Gw$ is continuous in the $\mathbb{C}$-topology.*
(3) *$\det_w$ is not a regular function on $\overline{Gw}$ if $m > 1$.*
(4) *$\overline{Gw}$ is not a normal variety if $m > 1$.*

(5) *For all highest weight vectors* $f \in \mathcal{O}(Gw)$ *we have* $(\det_w)^k f \in \mathcal{O}(\overline{Gw})$ *for some* $k \in \mathbb{Z}$.

We can also show a variant of this result with det replaced by $\det^2$. This is of interest since $(\det g)^2 = 1$ for all $g \in \mathrm{stab}(\langle m \rangle)$, cf. Proposition 4.1.

Corollary 7.2. *(1) The orbit closure of the matrix multiplication tensor* $\langle n, n, n \rangle$ *is not normal if* $n > 1$. *(2) The orbit closure of the unit tensor* $\langle m \rangle$ *is not normal if* $m \geq 5$.

The nonnormality of these orbit closures indicates that the extension problem is delicate. Kumar [22] recently obtained similar conclusions for the orbit closures of the determinant and permanent by different methods.

We also make the following general observation.

Proposition 7.3. *Suppose that* $w \in W$ *is stable. Then* $\mathrm{stab}(w)$ *is reductive,* Gw *is affine. Further,* $\overline{Gw} \setminus Gw$ *is either empty or of pure codimension one in* $\overline{Gw}$.

8. Moment polytopes

Since the semigroups $S(w)$ seem hard to determine, one may take a coarser viewpoint, as already suggested by Strassen [38, Eq. (57)]. We set $\Delta_{\underline{m}} := \Delta_{m_1} \times \Delta_{m_2} \times \Delta_{m_3}$, where $\Delta_m := \{x \in \mathbb{R}^m \mid x_1 \geq \ldots \geq x_m \geq 0, \sum_i x_i = 1\}$.

Definition 8.1. The *moment polytope* $P(w)$ of a tensor $w \in W$ is defined as the closure of the set $\left\{\frac{1}{d}\underline{\lambda} \mid d > 0, \underline{\lambda} \in S(w) \cap \Lambda_d^+(\underline{m})\right\}$.

Note that $P(w) \subseteq \Delta_{\underline{m}}$ is a polytope since $S(w)$ is a finitely generated semigroup. We have

$$\overline{Gw} \subseteq \overline{Gv} \Longrightarrow S(w) \subseteq S(v) \Longrightarrow P(w) \subseteq P(v)$$

Hence exhibiting some point in $P(w) \setminus P(\langle m \rangle)$ would establish the lower bound $\underline{R}(w) > m$.

The moment polytope of a generic tensor w of format $\underline{m}$ equals the *Kronecker polytope* $P(\underline{m})$, which is defined as the closure of $\{\frac{1}{d}\underline{\lambda} \mid \underline{\lambda} \in K(\underline{m}), \underline{\lambda} \in \Lambda_d^+(\underline{m})\}$, compare Lemma 3.2. This complicated polytope has been the object of several recent investigations [1, 20, 32, 4] and $P(\underline{m})$ is by now understood to a certain extent. We remark that the Kronecker polytope $P(\underline{m})$ is closely related to the quantum marginal problem of quantum information theory, cf. [7, 20].

Let $u_m := (1/m, \ldots, 1/m) \in \Delta_m$ denote the uniform distribution and put $u_{\underline{m}} := (u_m, u_m, u_m)$. The following follows, e.g., from [38, Satz 11].

Lemma 8.2. *We have* $u_{\underline{m}} \in P(w)$ *both for* $w = \langle m \rangle$ *and* $w = \langle n, n, n \rangle$, $m = n^2$.

Resolving the following question seems of great relevance.

Problem 8.3. Determine the moment polytopes of unit tensors and matrix multiplication tensors.

Replacing $S(w)$ by $S^o(w)$ in the definition of $P(w)$ we obtain the larger polytope $P^o(w)$.

Theorem 8.4. *We have* $P^o(\langle m \rangle)) = \Delta(m, m, m)$ *and* $P^o(\langle n, n, n \rangle)) = \Delta(n^2, n^2, n^2)$.

The statement for the unit tensors is an easy consequence of Corollary 4.5(2). The second statement relies on Theorem 5.3 and [4].

Lemma 8.5. *Let* w *be stable and suppose that* $u_{\underline{m}} \in P(w)$. *Then there exists* $\delta > 0$ *such that for all* $\underline{x} \in P^o(w)$ *and all* $0 \leq t \leq \delta$ *we have* $t\underline{x} + (1-t)u_{\underline{m}} \in P(w)$.

By combining Theorem 8.4 with Lemma 8.2, Lemma 8.5, and the stability of the unit and matrix multiplication tensors, we obtain the following result.

Corollary 8.6. *For both $w = \langle m\rangle$ and $w = \langle n, n, n\rangle$, $m = n^2$, $u_{\underline{m}}$ is an interior point of $P(w)$ relative to the affine hull of $\Delta_{\underline{m}}$. In particular, $\dim P(w) = \dim \Delta_{\underline{m}}$.*

9. Outline of some proofs

9.1. **Proof of Theorem 4.4.** Let $\underline{\lambda} \in \Lambda_d(m, m, m)$. The weight decomposition

$$V_{\underline{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} = \bigoplus_{\alpha,\beta,\gamma} V^{\alpha}_{\lambda_1} \otimes V^{\beta}_{\lambda_2} \otimes V^{\gamma}_{\lambda_3}$$

yields $(V_{\underline{\lambda}})^{D_m} = \bigoplus_{\alpha,\beta,\gamma} \big(V^{\alpha}_{\lambda_1} \otimes V^{\beta}_{\lambda_2} \otimes V^{\gamma}_{\lambda_3}\big)^{D_m}$, cf. Proposition 4.1. We claim that

$$\big(V^{\alpha}_{\lambda_1} \otimes V^{\beta}_{\lambda_2} \otimes V^{\gamma}_{\lambda_3}\big)^{D_m} = \begin{cases} V^{\alpha}_{\lambda_1} \otimes V^{\alpha}_{\lambda_2} \otimes V^{\alpha}_{\lambda_3} & \text{if } \alpha = \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let $v \in (V^{\alpha}_{\lambda_1} \otimes V^{\beta}_{\lambda_2} \otimes V^{\gamma}_{\lambda_3})^{D_m}$ be nonzero. For $t = (\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c) \in D_m$ we obtain $v = tv = a^{\alpha} b^{\beta} c^{\gamma} v = a^{\alpha-\gamma} b^{\beta-\gamma} v$, using $a_i b_i c_i = 1$. Since the $a_i, b_i \in \mathbb{C}^{\times}$ are arbitrary, we infer $\alpha = \beta = \gamma$. The argument can be reversed.

We put now

$$A := \big\{\alpha \in \mathbb{Z}^m \;\big|\; |\alpha| = d,\, \alpha \preceq \lambda_1 \curlywedge \lambda_2 \curlywedge \lambda_3\big\}, \quad M^{\alpha} := V^{\alpha}_{\lambda_1} \otimes V^{\alpha}_{\lambda_2} \otimes V^{\alpha}_{\lambda_3}$$

and note that $M^{\alpha} \neq 0$ for all $\alpha \in A$. We have just seen that $(V_{\underline{\lambda}})^{D_m} = \oplus_{\alpha\in A} M^{\alpha}$.

The set A is invariant under the S_m-action and its orbits intersect $\mathsf{Par}_m(d)$ in exactly one partition. We note that $\pi M^{\alpha} = M^{\pi\alpha}$ for $\pi \in S_m$. Let $\mathcal{B}$ denote the set of orbits and put $M_B := \oplus_{\alpha\in B} M^{\alpha}$ for $B \in \mathcal{B}$. Then $(V_{\underline{\lambda}})^{D_m} = \oplus_{B\in\mathcal{B}} M_B$. Proposition 4.1 tells us $H_m = D_m S_m$ and hence

$$(V_{\underline{\lambda}})^{H_m} = \big((V_{\underline{\lambda}})^{D_m}\big)^{S_m} = \bigoplus_{B\in\mathcal{B}} (M_B)^{S_m}$$

using that the M_B are S_m-invariant. In order to complete the proof it suffices to show that

$$\dim(M_B)^{S_m} = \dim(M^{\alpha})^{\mathrm{stab}(\alpha)} \quad \text{for } B = S_m\alpha,\ \alpha \in A \cap \mathsf{Par}_m(d).$$

For proving this, we fix $\alpha \in A \cap \mathsf{Par}_m(d)$ and write $H := \mathrm{stab}(\alpha)$. Let $\pi_1, \ldots, \pi_t$ be a system of representatives for the left cosets of H in S_m with $\pi_1 = \mathrm{id}$. So $S_m = \pi_1 H \cup \cdots \cup \pi_t H$. Then the S_m-orbit of α equals $S_m\alpha = \{\pi_1\alpha, \ldots, \pi_t\alpha\}$. Consider

$$M_B = \bigoplus_{j=1}^{t} \pi_j M^{\alpha}$$

and the corresponding projection $p\colon M_B \to M^{\alpha}$. Suppose that $v = \sum_j v_j \in (M_B)^{S_m}$ with $v_j \in \pi_j M^{\alpha}$. Since the spaces $\pi_1 M^{\alpha}, \ldots, \pi_t M^{\alpha}$ are permuted by the action of S_m, we derive from $v = \pi_k v = \sum_j \pi_k v_j$ that $v_j = \pi_j v_1$. Moreover, since $\sigma \in H$ fixes M^{α} and permutes the spaces $\pi_2 M^{\alpha}, \ldots, \pi_t M^{\alpha}$, we obtain $\sigma v_1 = v_1$. Therefore, $(M_B)^{S_m} \to (M^{\alpha})^H, v \mapsto p(v) = v_1$ is well defined and injective. We claim that this map is also surjective.

For showing this, let $v_1 \in (M^{\alpha})^H$, set $v_j := \pi_j v_1$, and put $v := \sum_j v_j$. Clearly, $p(v) = v_1$. Fix $\sigma \in H$ and i. For any j there is a unique $k = k(j)$ such that $\sigma\pi_i\pi_j H = \pi_k H$. Moreover,

$j \mapsto k(j)$ is a permutation of $\{1,\dots,t\}$. Using the H-invariance of v_1 we obtain that $\sigma\pi_i v_j = \sigma\pi_i\pi_j v_1 = \pi_k v_1 = v_k$. Therefore $\sigma\pi_i v = \sum_k v_k = v$. Thus $v \in (M_B)^{S_m}$. □

9.2. **Proof of Theorem 5.3.** The group morphisms

$$\begin{aligned}
\Gamma_{12}\colon \mathrm{GL}(U_1^*)\times\mathrm{GL}(U_2) &\to \mathrm{GL}(U_1^*\otimes U_2), \quad (a^*,b)\mapsto a^*\otimes b\\
\Gamma_{23}\colon \mathrm{GL}(U_2^*)\times\mathrm{GL}(U_3) &\to \mathrm{GL}(U_2^*\otimes U_3), \quad (b^*,c)\mapsto b^*\otimes c\\
\Gamma_{31}\colon \mathrm{GL}(U_3^*)\times\mathrm{GL}(U_1) &\to \mathrm{GL}(U_3^*\otimes U_1), \quad (c^*,a)\mapsto c^*\otimes a
\end{aligned}$$

combine to a morphism $\Gamma\colon \Pi\to G$, where Π denotes the group

$$\Pi := \mathrm{GL}(U_1^*)\times\mathrm{GL}(U_2)\times\mathrm{GL}(U_2^*)\times\mathrm{GL}(U_3)\times\mathrm{GL}(U_3^*)\times\mathrm{GL}(U_1).$$

Moreover, we have the group morphisms

$$\Lambda_i\colon \mathrm{GL}(U_i)\to\mathrm{GL}(U_i^*)\times\mathrm{GL}(U_i), a_i\mapsto((a_i^*)^{-1},a_i)$$

combining to a morphism (note the permutation)

$$\Lambda\colon K\to\Pi,\ (a_1,a_2,a_3)\mapsto\big((a_1^*)^{-1}),a_2,(a_2^*)^{-1},a_3,(a_3^*)^{-1},a_1\big).$$

We have thus factored the morphism $\Phi\colon K\to G$ as $\Phi=\Gamma\circ\Lambda$, cf. (5.1). Proposition 5.1 states that $H=\mathrm{im}\Phi$. In order to determine $\dim(V_{\underline{\lambda}})^H$, we first describe the splitting of $V_{\underline{\lambda}}$ into irreducible Π-modules with respect to Γ and then, in a second step, extract their K-invariants.

For the first step, note that, upon restriction with respect to Γ_{12}, we have the decomposition

$$V_{\lambda_{12}}(\mathrm{GL}(U_1^*\otimes U_2)) = \bigoplus_{\mu_1,\tilde\mu_2} g(\lambda_{12},\mu_1,\tilde\mu_2)V_{\mu_1}(\mathrm{GL}(U_1^*))\otimes V_{\tilde\mu_2}(\mathrm{GL}(U_2)),$$

where the sum is over all partitions $\mu_1\vdash_{n_1} d$, $\tilde\mu_2\vdash_{n_2} d$. For this characterization of the Kronecker coefficients g see [34, (7.221), p. 537]. Similarly,

$$V_{\lambda_{23}}(\mathrm{GL}(U_2^*\otimes U_3)) = \bigoplus_{\mu_2,\tilde\mu_3} g(\lambda_{23},\mu_2,\tilde\mu_3)V_{\mu_2}(\mathrm{GL}(U_2^*))\otimes V_{\tilde\mu_3}(\mathrm{GL}(U_3)),$$

$$V_{\lambda_{31}}(\mathrm{GL}(U_3^*\otimes U_1)) = \bigoplus_{\mu_3,\tilde\mu_1} g(\lambda_{31},\mu_3,\tilde\mu_1)V_{\mu_3}(\mathrm{GL}(U_3^*))\otimes V_{\tilde\mu_1}(\mathrm{GL}(U_1)),$$

where the sums are over all $\mu_2\vdash_{n_2} d$, $\tilde\mu_3\vdash_{n_3} d$ and $\mu_3\vdash_{n_3} d$, $\tilde\mu_1\vdash_{n_1} d$, respectively. This describes the splitting of $V_{\underline{\lambda}}$ into irreducible Π-modules with respect to Γ.

For the second step we note that $V_{\mu_i}(\mathrm{GL}(U_i^*))\simeq V_{\mu_i^*}(\mathrm{GL}(U_i))$, when we view the left hand side as a $\mathrm{GL}(U_i)$-module via the isomorphism $\mathrm{GL}(U_i)\to\mathrm{GL}(U_i^*), a_i\mapsto(a_i^*)^{-1}$.

As a consequence of the Littlewood-Richardson rule [34, 11] we obtain (compare [11, Eq. (11), p.149])

$$\dim\big(V_{\mu_i^*}(\mathrm{GL}(U_i))\otimes V_{\tilde\mu_i}(\mathrm{GL}(U_i))\big)^{\mathrm{GL}(U_i)} = \begin{cases}1 & \text{if } \mu_i=\tilde\mu_i,\\ 0 & \text{otherwise.}\end{cases} \tag{9.1}$$

We conclude that

$$\dim(V_{\underline{\lambda}})^H = \bigoplus_{\mu_1,\mu_2,\mu_3} g(\lambda_{12},\mu_1,\mu_2)\,g(\lambda_{23},\mu_2,\mu_3)\,g(\lambda_{31},\mu_3,\mu_1)$$

as claimed. □

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Appendix: Section 3

10.1. **Highest weight vectors.** For the following general facts see [14, 17, 21]. In general, let G be a reductive group and fix a Borel subgroup with corresponding maximal unipotent subgroup U and maximal torus T. Let M be a rational G-module. By a *highest weight vector* in M of weight $\lambda \in \Lambda_G^+$ we understand a U-invariant weight vector in M of weight λ. These vectors (including the zero vector) form a linear subspace of M that we shall denote by $\mathcal{H}_\lambda(M)$. It is known that M is irreducible iff $\mathcal{H}_\lambda(M)$ is one-dimensional. If $\varphi\colon M \to N$ is a surjective G-module morphism, then $\varphi(\mathcal{H}_\lambda(M)) = \mathcal{H}_\lambda(N)$.

10.2. **Schur-Weyl duality.** For the following known facts see [12]. Recall that $[\lambda]$ denotes the irreducible module of the symmetric group S_d associated with a partition $\lambda \vdash d$. Let V denote a vector space of dimension m. The group S_d acts on $V^{\otimes d}$ by permutation. We define the λth *Schur-Weyl module* by

$$S_\lambda(V) := \mathrm{Hom}_{S_d}([\lambda], V^{\otimes d}).$$

Note that $S_\lambda(V)$ becomes a $\mathrm{GL}(V)$-module in a natural way. It is well known that $S_\lambda(V) = 0$ if λ has more than m parts. Otherwise, $S_\lambda(V)$ is an irreducible $\mathrm{GL}(V)$-module, and all the irreducible $\mathrm{GL}(V)$-modules of degree d are isomorphic to $S_\lambda(V)$ for some $\lambda \vdash_m d$.

A linear map $\varphi\colon V \to W$ induces the linear map $S_\lambda(\varphi)\colon S_\lambda V \to S_\lambda W, \alpha \mapsto \varphi^{\otimes d}\alpha$ and the functorial property $S_\lambda(\psi\phi) = S_\lambda(\psi)S_\lambda(\phi)$ clearly holds.

We have natural injective maps $[\lambda] \otimes S_\lambda(V) \to V^{\otimes d}, x \otimes \alpha \mapsto \alpha(x)$, which yield the following canonical injective morphism of $S_d \times \mathrm{GL}(V)$-modules:

$$\bigoplus_{\lambda \vdash_m d} [\lambda] \otimes S_\lambda(V) \to V^{\otimes d}. \tag{10.1}$$

Schur-Weyl duality states that this map is surjective. In particular, (10.1) yields the isotypic decomposition of $V^{\otimes d}$ with respect to the action of $S_d \times \mathrm{GL}(V)$.

Depending on the context, we also use the notation $V_\lambda(\mathrm{GL}(V)) := S_\lambda(V)$ for emphasizing the dependence on the group.

10.3. **Decomposition of $\mathcal{O}(W)$.** Let W_i be a vector space of dimension m_i for $i = 1, 2, 3$, put $W := W_1 \otimes W_2 \otimes W_3$, and $G = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3)$. Applying Schur-Weyl duality (10.1) to W_i^* and taking the tensor product yields the isotypic decomposition with respect the action of of the group $S_d \times S_d \times S_d \times G$:

$$\begin{aligned}(W^*)^{\otimes d} &\simeq (W_1^* \otimes W_2^* \otimes W_3^*)^{\otimes d} \simeq (W_1^*)^{\otimes d} \otimes (W_2^*)^{\otimes d} \otimes (W_3^*)^{\otimes d} \\ &\simeq \bigoplus_{\underline{\lambda} \in \Lambda_d^+(\underline{m})} [\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3] \otimes S_{\underline{\lambda}}(W^*),\end{aligned}$$

where we have put $S_{\underline{\lambda}}(W^*) := S_{\lambda_1}(W_1^*) \otimes S_{\lambda_2}(W_2^*) \otimes S_{\lambda_3}(W_3^*)$, which is the same as $V_{\underline{\lambda}}(G)^*$ since $S_{\lambda_i}(W_i^*) \simeq (S_{\lambda_i}(W_i))^*$. We interpret S_d as a subgroup of $S_d \times S_d \times S_d$ with respect to the diagonal embedding $\pi \mapsto (\pi, \pi, \pi)$ and note that

$$\mathcal{O}(W)_d = \mathsf{Sym}^d W^* = \big((W^*)^{\otimes d}\big)^{S_d}.$$

We therefore arrive at the following canonical isomorphism of G-modules:

$$\mathcal{O}(W)_d = \mathsf{Sym}^d W^* \simeq \bigoplus_{\underline{\lambda}\in\Lambda_d^+(\underline{m})} \big([\lambda_1]\otimes[\lambda_2]\otimes[\lambda_3]\big)^{S_d} \otimes S_{\underline{\lambda}}(W^*). \tag{10.2}$$

In particular, we obtain

$$g(\underline{\lambda}) = \dim\big([\lambda_1]\otimes[\lambda_2]\otimes[\lambda_3]\big)^{S_d} = \mathrm{mult}(V_{\underline{\lambda}}(G)^*, \mathcal{O}(W)_d) \tag{10.3}$$

by the definition of Kronecker coefficients.

10.4. **Proof of Lemma 3.2.** The restriction $\mathrm{res}_{d,w}\colon \mathcal{O}(W)_d \to \mathcal{O}(\overline{Gw})_d$ is a surjective morphism of G-modules, for each degree $d\in\mathbb{N}$. Hence $S(w)\subseteq K(\underline{m})$ follows from (10.3).

Let $\underline{\lambda}_1,\dots,\underline{\lambda}_s$ be a list of generators of the semigroup $K(\underline{m})$. By (10.3), there exists a highest weight vector $F_i\in\mathcal{O}(W)_{d_i}$ of weight $\underline{\lambda}_i^*$, for each i. Assume $F_1(w)\cdots F_s(w)\neq 0$. Then $\mathrm{res}_{d_i,w}(F_i)$ is a highest weight vector of weight $\underline{\lambda}_i^*$ in $\mathcal{O}(\overline{Gw})_{d_i}$, hence $\underline{\lambda}_i\in S(w)$. Since the generators $\underline{\lambda}_1,\dots,\underline{\lambda}_s$ of $K(\underline{m})$ are all contained in $S(w)$, we get $K(\underline{m})\subseteq S(w)$. So we have shown that the nonempty open set $\{w\in W \mid F_1(w)\cdots F_s(w)\neq 0\}$ is contained in $\{w\in W\mid K(\underline{m})\subseteq S(w)\}$.

10.5. **Inheritance.** For this section compare [24, 6]. We first recall our standard notations: Let $W := W_1\otimes W_2\otimes W_3$ and $W' := W_1'\otimes W_2'\otimes W_3'$ be of format $\underline{m}$ and $\underline{m}'$, respectively. Let $\iota_i\colon W_i\hookrightarrow W_i'$ be inclusions of vector spaces and choose linear projections $p_i\colon W_i'\to W_i$ such that $p_i\,\iota_i = \mathrm{id}_{W_i}$. For $\underline{\lambda}\in\Lambda_d^+(\underline{m}')$ the dual maps $\iota_i^*\colon W_i'^*\to W_i^*$ induce maps $S_{\lambda_i}(\iota_i^*)\colon S_{\lambda_i}(W_i'^*)\to S_{\lambda_i}(W_i^*)$. These maps are surjective since by the functoriality we have $S_{\lambda_i}(\iota_i^*)\,S_{\lambda_i}(p_i^*)=\mathrm{id}$. We further set $\iota:=\iota_1\otimes\iota_2\otimes\iota_3$, $p:=p_1\otimes p_2\otimes p_3$, and $S_{\underline{\lambda}}(\iota^*) := S_{\lambda_1}(\iota_1^*)\otimes S_{\lambda_2}(\iota_2^*)\otimes S_{\lambda_3}(\iota_3^*)$. Note that $S_{\underline{\lambda}}(\iota^*)\colon S_{\underline{\lambda}}(W'^*)\to S_{\underline{\lambda}}(W*)$ is surjective.

Let $\mathrm{res}_d\colon \mathcal{O}(W')_d\to\mathcal{O}(W)_d$ denote the restriction of regular functions. Equation (10.2) yields the following commutative diagram (writing equalities for canonical isomorphisms)

$$\begin{array}{ccc} \mathcal{O}(W')_d & = & \bigoplus_{\underline{\lambda}\in\Lambda_d^+(\underline{m}')} \;\Sigma_{\underline{\lambda}}\otimes S_{\underline{\lambda}}(W'^*) \\ {\scriptstyle \mathrm{res}_d}\downarrow & & \qquad\qquad {\scriptstyle \mathrm{id}\otimes S_{\underline{\lambda}}(\iota^*)}\downarrow \\ \mathcal{O}(W)_d & = & \bigoplus_{\underline{\lambda}\in\Lambda_d^+(\underline{m})} \;\Sigma_{\underline{\lambda}}\otimes S_{\underline{\lambda}}(W^*), \end{array} \tag{10.4}$$

with the vector spaces $\Sigma_{\underline{\lambda}} := \big([\lambda_1]\otimes[\lambda_2]\otimes[\lambda_3]\big)^{S_d}$. Note that the vertical arrows are surjective.

Recall that $G:=\mathrm{GL}(W_1)\times\mathrm{GL}(W_2)\times\mathrm{GL}(W_3)$ and $G':=\mathrm{GL}(W_1')\times\mathrm{GL}(W_2')\times\mathrm{GL}(W_3')$ and let $I(Z)$ denote the vanishing ideal of an affine variety Z. We notationally identify $w\in W$ with it image w' under the inclusion $\iota\colon W\to W'$.

Lemma 10.1. *1. For $w\in W$ we have $p(\overline{G'w})=\overline{Gw}$.*

2. We have $\mathrm{res}(I(\overline{G'w}))=I(\overline{Gw})$ for the restriction $\mathrm{res}\colon\mathcal{O}(W')\to\mathcal{O}(W)$.

Proof. 1. Put $F_i:=\ker p_i$ so that $W_i'=W_i\oplus F_i$ and p_i is the projection onto W_i along F_i. Let $g'=(g_1',g_2',g_3')\in G'$ and consider $g_i:=p_i\,g_i'\iota_i$. Note that $g_i\colon W_i\to W_i$ is a linear map that may be noninvertible. It is easy to check that $p(g'w)=(g_1\otimes g_2\otimes g_3)w$. This implies the first assertion $p(\overline{G'w})=\overline{Gw}$.

2. For $f\in I(\overline{Gw})$ define $F:=f\,p\in\mathcal{O}(W')$. Then $\mathrm{res}(F)=f$ and part (1) implies that F vanishes on $\overline{G'w}$. This shows $\mathrm{res}(I(\overline{G'w}))\supseteq I(\overline{Gw})$. The other inclusion is obvious. □

We analyze now the G'-submodule $I(\overline{G'w})_d \subseteq \mathcal{O}(W')_d$. The isotypic decomposition (10.2) implies that for each $\underline{\lambda} \in \Lambda_d^+(\underline{m}')$ there exists a linear subspace $J_{\underline{\lambda}}(w) \subseteq \Sigma_{\underline{\lambda}}$ such that

$$I(\overline{G'w})_d \;=\; \bigoplus_{\underline{\lambda}\in\Lambda_d^+(\underline{m}')} J_{\underline{\lambda}}(w) \otimes S_{\underline{\lambda}}(W'^*).$$

By Lemma 10.1(2) and the commutative diagram (10.4) we have

$$I(\overline{Gw})_d = \mathrm{res}_d(I(\overline{Gw'})_d) \;=\; \bigoplus_{\underline{\lambda}\in\Lambda_d^+(\underline{m})} J_{\underline{\lambda}}(w) \otimes S_{\underline{\lambda}}(W^*).$$

Since $\mathcal{O}(\overline{Gw})_d = \mathcal{O}(W)_d/I(\overline{Gw})_d$, this implies, for $\underline{\lambda} \in \Lambda_d^+(\underline{m})$,

$$\mathrm{mult}(S_{\underline{\lambda}}(W^*), \mathcal{O}(\overline{Gw})) = \dim \Sigma_{\underline{\lambda}} - \dim J_{\underline{\lambda}}(w) = \mathrm{mult}(S_{\underline{\lambda}}(W'^*), \mathcal{O}(\overline{G'w})). \tag{10.5}$$

The case $\underline{\lambda} \in \Lambda_d^+(\underline{m}') \setminus \Lambda_d^+(\underline{m})$ is covered by the following lemma.

Lemma 10.2. *We have* $J_{\underline{\lambda}}(w) = \Sigma_{\underline{\lambda}}$ *and hence* $\mathrm{mult}(S_{\underline{\lambda}}(W'^*), \mathcal{O}(\overline{G'w})) = 0$ *if* $\underline{\lambda} \in \Lambda_d^+(\underline{m}') \setminus \Lambda_d^+(\underline{m})$.

Proof. Let $M \subseteq \mathcal{O}(W')_d$ be a submodule such that $M \simeq S_{\underline{\lambda}}(W'^*)$. We need to show that $M \subseteq I(\overline{G'w})$ if $\underline{\lambda} \in \Lambda_d^+(\underline{m}') \setminus \Lambda_d^+(\underline{m})$.

It is convenient to prove the contraposition. So suppose that M is not contained in $I(\overline{G'w})$. Then there exists $F \in M$ such that $F(w) \neq 0$. (Indeed, by assumption there exist $F \in M$ and $g \in G'$ such that $0 \neq F(g^{-1}w) = (gF)(w)$. Just replace F by $gF \in M$.)

We may assume that F is a weight vector and let $-\underline{\alpha} = -(\alpha_1, \alpha_2, \alpha_3)$ denote its weight. Write $\alpha_i = (\alpha_{i1}, \dots, \alpha_{im_i'})$. Let $t = (t_1, t_2, t_3) \in G'$ with $t_i = \mathrm{diag}(t_{i1}, \dots, t_{im_i'})$ such that $t_{ij} = 1$ for $1 \leq j \leq m_i$. Then $tw = w$ and we have

$$F(w) = F(tw) = (t^{-1}F)(w) = \textstyle\prod_{i=1}^3 \prod_{j=m_i+1}^{m_i'} t_{ij}^{\alpha_{ij}} F(w).$$

Since $t_{ij} \in \mathbb{C}^\times$ is arbitrary for $j > m_i$, it follows that $\alpha_{ij} = 0$ for $j > m_i$.

Since $\underline{\alpha}$ is a weight of $M^* \simeq S_{\underline{\lambda}}(W')$ and $\underline{\lambda}$ is the highest weight of M, we have $\underline{\alpha} \preceq \underline{\lambda}$. From $\alpha_{ij} = 0$ for $j > m_i$ it follows that λ_i has at most m_i parts. This means $\underline{\lambda} \in \Lambda_d^+(\underline{m})$. ☐

Equation (10.5) and Lemma 10.2 imply Proposition 3.3.

10.6. **Decomposition of ring of regular functions on orbits.** Consider the action of $G \times G$ on G defined by $(g_1, g_2) \cdot g := g_1\, g\, g_2^{-1}$. This induces the following action of $G \times G$ on $\mathcal{O}(G)$:

$$((g_1, g_2) \cdot f)(g) := f(g^{-1}\, g\, g_2) \quad \text{where } g_1, g_2, g \in G, f \in \mathcal{O}(G).$$

The usual "left action" of G on $\mathcal{O}(G)$ is obtained by the embedding $G \hookrightarrow G \times G$, $g_1 \mapsto (1, \mathrm{id})$. But note that we also have a "right action" of G given by $G \hookrightarrow G \times G$, $g_2 \mapsto (1, g_2)$.

We now state the fundamental algebraic Peter Weyl Theorem for the group G, cf. [21, p. 93] or [14]. (This result actually holds for any reductive group G.)

Theorem 10.3. *The isotypic decomposition of* $\mathcal{O}(G)$ *as a* $G \times G$*-module is given as*

$$\mathcal{O}(G) = \bigoplus_{\underline{\lambda}\in\Lambda_G^+} V_{\underline{\lambda}}(G)^* \otimes V_{\underline{\lambda}}(G).$$

Here the $G \times G$-module structure on $V_{\underline{\lambda}}(G)^* \otimes V_{\underline{\lambda}}(G)$ is to be interpreted as $(g_1, g_2)(\ell \otimes v) := g_1\ell \otimes g_2 v$ for $\ell \in V_{\underline{\lambda}}(G)^*$ and $v \in V_{\underline{\lambda}}(G)$.

10.7. **Proof of Proposition 3.4.** The stabilizer H of w is a closed subgroup of G. It is possible to give $G/H = \{gH \mid g \in G\}$ the structure of an algebraic variety such that $G \to G/H$ is a morphism of varieties satisfying the universal property of quotients (cf. [17, Chap. 12]. This implies that G/H is isomorphic to Gw as a variety and this morphism is G-equivariant.

The morphism of coordinate rings $\mathcal{O}(G/H) \to \mathcal{O}(G)$ induced by $G \to G/H$ is injective, and maps to the subring

$$\mathcal{O}(G)^H = \{f \in \mathcal{O}(G) \mid \forall g \in G, h \in H \ f(gh) = f(g)\}$$

of H-invariant functions with respect to the right action of H on $\mathcal{O}(G)$. We note that $\mathcal{O}(G)^H$ is a G-submodule of $\mathcal{O}(G)$ with respect to the left action of G. Moreover, $\mathcal{O}(G/H) \to \mathcal{O}(G)^H$ is G-equivariant. The universal property of quotients implies the surjectivity of $\mathcal{O}(G/H) \to \mathcal{O}(G)^H$. So we have shown that $\mathcal{O}(Gw)$ is isomorphic to the G-module $\mathcal{O}(G)^H$.

Theorem 10.3 implies that

$$\mathcal{O}(G)^H = \bigoplus_{\underline{\lambda} \in \Lambda_G^+} V_{\underline{\lambda}}(G)^* \otimes V_{\underline{\lambda}}(G)^H.$$

Hence $\mathrm{mult}(V_{\underline{\lambda}}(G)^*, \mathcal{O}(G)^H) = \dim V_{\underline{\lambda}}(G)^H$, which completes the proof. □

Appendix: Section 4

10.8. **Stabilizers of associative algebras.** Let $\mathrm{Bil}(U, V; W)$ denote the space of bilinear maps $U \times V \to W$, where U, V, W are finite dimensional vector spaces. The group $G = \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on $\mathrm{Bil}(U, V; W)$ via $(\alpha, \beta, \gamma) \cdot \varphi := \gamma\, \varphi(\alpha^{-1} \times \beta^{-1})$. By definition, $\mathrm{GL}(U)$ acts on the dual module U^* via $\alpha \cdot \ell := (\alpha^{-1})^*(\ell)$ for $\alpha \in \mathrm{GL}(U)$, $\ell \in U^*$. It is straightforward to check that the canonical isomorphism $U^* \otimes V^* \otimes W \to \mathrm{Bil}(U, V; W)$ is G-equivariant. Hence we obtain the following result.

Lemma 10.4. *Let $\varphi \in \mathrm{Bil}(U, V; W)$ and $w \in U^* \otimes V^* \otimes W$ be the corresponding tensor. Then*

$$\mathrm{stab}(w) = \big\{((\alpha^{-1})^*, (\beta^{-1})^*, \gamma) \mid \forall u, v \ \varphi(\alpha(u), \beta(v)) = \gamma(\varphi(u, v))\big\}.$$

Now let A be a finite dimensional associative $\mathbb{C}$-algebra with 1. Its multiplication map $A \times A \to A$ corresponds to a tensor $w_A \in A^* \otimes A^* \otimes A$. We denote by $A^\times$ the unit group of A and by $\mathrm{Aut}A$ its group of algebra automorphisms. For $a \in A$ we denote by $L_a \colon A \to A, x \mapsto ax$ the left multiplication with a. Similarly, R_a denotes the right multiplication with a.

The following observation goes back to [10].

Lemma 10.5. *We have*

$$\mathrm{stab}(w_A) = \Big\{\big(L^*_{\varepsilon^{-1}}(\psi^{-1})^*, R^*_{\eta^{-1}}(\psi^{-1})^*, L_\varepsilon R_\eta \psi\big) \ \Big| \ \varepsilon, \eta \in A^\times, \psi \in \mathrm{Aut}A\Big\}.$$

Proof. Let $\alpha, \beta, \gamma \in \mathrm{GL}(A)$. Suppose that $\big((\alpha^{-1})^*, (\beta^{-1})^*, \gamma\big) \in \mathrm{stab}(w_A)$. By Lemma 10.4 we have $\alpha(a)\beta(b) = \gamma(ab)$ for all $a, b \in A$. Plugging in 1 we get $\alpha(a)\beta(1) = \gamma(a)$ and $\alpha(1)\beta(b) = \gamma(b)$. Hence $\varepsilon := \alpha(1)$ and $\eta := \beta(1)$ must be units of A. We define now $\psi(a) := \varepsilon^{-1}\gamma(a)\eta^{-1}$. Then we have $\psi(1) = 1$ and

$$\psi(a)\psi(b) = \varepsilon^{-1}\gamma(a)\eta^{-1}\,\varepsilon^{-1}\gamma(b)\eta^{-1} = \varepsilon^{-1}\alpha(a)\beta(b)\eta^{-1} = \varepsilon^{-1}\gamma(ab)\eta^{-1} = \psi(ab).$$

Therefore $\psi \in \mathrm{Aut}A$. By construction, $\alpha = L_\varepsilon\psi$, $\beta = R_\eta\psi$, and $\gamma = L_\varepsilon R_\eta\psi$, and hence $(\alpha^{-1})^* = L^*_{\varepsilon^{-1}}(\psi^{-1})^*$, $(\beta^{-1})^* = R^*_{\eta^{-1}}(\psi^{-1})^*$. The argument is reversible. $\square$

10.9. **Proof of Proposition 4.1.** Let S_m denote the diagonal embedding of the symmetric group in G_m. Obviously, $D_m \cap S_m = \{\mathrm{id}\}$. It is easy to see that S_m normalizes D_m. Hence $D_m S_m$ is a subgroup of G_m and D_m is a normal divisor of $D_m S_m$. It remains to prove that the stabilizer H_m equals $D_m S_m$. The inequality $D_m S_m \subseteq H_m$ is obvious.

Note that $\langle m\rangle$ is the structural tensor of the algebra $A = \mathbb{C}^m$. It is straightforward to check that $\mathrm{Aut}A = \{P_\pi \mid \pi \in S_m\}$. Note that $(P_\pi^{-1})^* = P_\pi$. Hence Lemma 10.5 implies

$$H_m = \mathrm{stab}(\langle m\rangle) = \left\{(\mathrm{diag}(\varepsilon^{-1})P_\pi, \mathrm{diag}(\eta^{-1})P_\pi, \mathrm{diag}(\varepsilon\eta)P_\pi) \mid \varepsilon, \eta \in (\mathbb{C}^\times)^m, \pi \in S_m\right\}$$

and we obtain $H_m = D_m S_m$. $\square$

Lemma 10.6. *If the stabilizer of $w \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ contains H_m, then $w = c\,\langle m\rangle$ for some $c \in \mathbb{C}$.*

Proof. Assume the stabilizer of $w = \sum w_{ijk} e_i \otimes e_j \otimes e_k$ contains H_m. By contradiction, we suppose that $w_{ijk} \neq 0$ for some i, j, k with $i \neq k$. For any $(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \in D_m$ we have $a_i b_j c_k w_{ijk} = w_{ijk}$ and hence $a_i b_j c_k = 1 = a_k b_k c_k$, which implies $a_i = a_k b_k / b_j$. However, defining $\tilde{a}_i = 2a_i$, $\tilde{c}_i = \frac{1}{2}c_i$, $\tilde{a}_\ell = a_\ell$, $\tilde{c}_\ell = c_\ell$ for $\ell \neq i$ we get $(\mathrm{diag}(\tilde{a}), \mathrm{diag}(b), \mathrm{diag}(\tilde{c})) \in D_m$. This yields the contradiction $\tilde{a}_i = \tilde{a}_k b_k / b_j = a_k b_k / b_j = a_i$. We have thus shown that $w_{ijk} \neq 0$ implies $i = k$. By symmetry, we conclude that $w_{ijk} = 0$ unless $i = j = k$. Finally, from the invariance of w under $S(m)$, we get $w_{iii} = w_{111}$ for all i. Hence $w = w_{111}\langle m\rangle$. $\square$

10.10. **Stability.** We need some criterion for testing stability. By a *one-parameter subgroup* of G_s we understand a morphism $\sigma\colon \mathbb{C}^\times \to G_s$ of algebraic groups. The *centralizer* $Z_{G_s}(R_s)$ of a subgroup R_s of G_s is defined as the set of $g \in G_s$ such that $gh = hg$ for all $h \in R_s$. For instance, let T_s denote the maximal torus of G_s. Then we have $Z_{G_s}(T_s) = T_s$.

The following important stability criterion is a consequence Kempf's [19] refinement of the Hilbert-Mumford criterion.

Theorem 10.7. *Let $w \in W$ be a tensor and R_s be a reductive subgroup of G_s contained in the stabilizer of w. We assume that for all one-parameter subgroups σ of G_s, with image in the centralizer $Z_{G_s}(R_s)$, the limit $\lim_{t\to 0}\sigma(t)w$ lies in the G_s-orbit of w, provided the limit exists. Then w is stable.*

Proof. Suppose that w is not stable. Then there is a nonempty closed G_s-invariant subset Y of $\overline{G_s w} \setminus G_s w$. Kempf's result [19] states that there exists a one-parameter subgroup $\sigma\colon \mathbb{C}^\times \to Z_{G_s}(R_s)$ such that $\lim_{t\to 0}\sigma(t)w \in Y$. Hence this limit does not lie in $G_s w$. $\square$

10.11. **Proof of Proposition 4.2.** We apply Theorem 10.7 with $R_s := D_m \cap G_s$. The group R_s is a torus and hence reductive [21]. It is easy to see that $Z_{G_s}(R_s)$ equals the maximal torus T_s of G_s.

Any one-parameter subgroup $\sigma\colon \mathbb{C}^\times \to T_s$ is of the form $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t))$ with

$$\sigma_1(t) = \mathrm{diag}(t^{\mu_i}),\ \ \sigma_2(t) = \mathrm{diag}(t^{\nu_i}),\ \ \sigma_3(t) = \mathrm{diag}(t^{\pi_i})$$

with integers μ_i, ν_i, π_i. Since $\det\sigma_k(t) = 1$ we must have

$$\sum_i \mu_i = 0, \ \sum_i \nu_i = 0, \ \sum_i \pi_i = 0.$$

We have $\sigma(t)\langle m\rangle = \sum_i t^{\mu_i+\nu_i+\pi_i} e_i \otimes e_i \otimes e_i$. If the limit of $\sigma(t)\langle m\rangle$ exists for $t \to 0$, then $\mu_i+\nu_i+\pi_i \geq 0$ for all i. On the other hand, $\sum_i(\mu_i+\nu_i+\pi_i) = 0$. It follows that $\mu_i+\nu_i+\pi_i = 0$ for all i, hence $\sigma(t)\langle m\rangle = \langle m\rangle$. $\square$

10.12. **Proof of Lemma 4.3.** Recall that a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of natural numbers such that only finitely many components are nonzero. We define the *length* of $\lambda \neq 0$ as $\mathsf{len}(\lambda) := \max\{i \mid \lambda_i \neq 0\}$ and we put $\mathsf{len}(0) := 0$. The componentwise sum of partitions is well defined. Note that $\lambda \preceq \mu$ implies $\lambda + \nu \preceq \mu + \nu$ for any partitions λ, μ, ν. We call $s(n) := (n, n-1, \dots, 1)$ the *symmetric staircase partition* with n rows and n columns. Note that if λ is a regular partition with at least j nonzero rows, then $\lambda - s(j)$ is again a partition, since λ has at least $j - i + 1$ boxes in row i.

(1) Let $d = qm + r$ with $0 \leq r < m$. Then $\square_m(d) := (q^m) + (1^r)$ (frequency notation) is the unique smallest element of $\mathsf{Par}_m(d)$. The corresponding diagram has q columns of length m plus one additional column of length r.

For given $d, m \in \mathbb{N}$ we set $\ell := \ell(m,d) := \max\{n \leq m \mid n(n+1)/2 \leq d\}$ and we define the *staircase partition* $\perp_m(d) := s(\ell) + \square_\ell(d - |s(\ell)|)$, see Figure 1. We observe that $\mathsf{len}(\perp_m(d)) = \ell$ and moreover $\perp_m(d) = \perp_\ell(d)$.

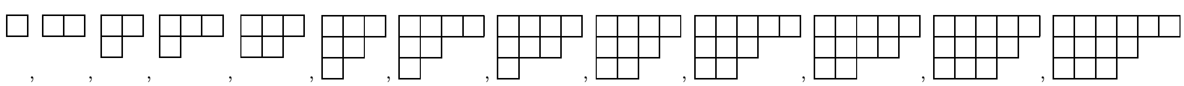

FIGURE 1. The staircase partitions $\perp_3(d)$ for $d = 1, \dots, 13$.

Part (1) of Lemma 4.3 claims that $\perp_m(d) \preceq \beta$ for any regular partition $\beta \in \mathsf{Par}_m(d)$.

For showing this, put $\tilde{\ell} := \mathsf{len}(\beta)$ and note that $\beta - s(\tilde{\ell})$ is a partition by the previous observation. If we had $\ell + 1 \leq \tilde{\ell}$, then $\tilde{\beta} - s(\ell+1)$ would be a partition as well and hence $d = |\beta| \geq \frac{(\ell+1)(\ell+2)}{2}$ contradicting the maximality of ℓ. So we have $\mathsf{len}(\perp_m(d)) = \ell \geq \tilde{\ell}$ and hence $\perp_m(d) - s(\tilde{\ell})$ is a partition.

We note that the subpartition consisting of the first $\tilde{\ell}$ rows of $\perp_m(d) - s(\tilde{\ell})$ equals $\square_{\tilde{\ell}}(d')$ for some $d' \leq d - |s(\tilde{\ell})|$. Moreover, $\square_{\tilde{\ell}}(d') \preceq \square_{\tilde{\ell}}(d - |s(\tilde{\ell})|) \preceq \beta - s(\tilde{\ell})$, where the last inequality is due to the minimality of $\square_{\tilde{\ell}}(\cdot)$. It follows that $\perp_m(d) - s(\tilde{\ell}) \preceq \beta - s(\tilde{\ell})$, which completes the proof of part (1).

(2). For $k > \frac{m(m+1)}{2} + m$ we define the following regular partition in $\mathsf{Par}_{m+1}(d + km)$:

$$\lambda^k_{\mathsf{reg}} := (\lambda_1 + k - 1, \lambda_2 + k - 2, \dots, \lambda_m + k - m, \frac{m(m+1)}{2}).$$

Part (1) yields $\perp_{m+1}(d+km) \preceq \lambda^k_{\mathsf{reg}}$. Since $\lambda^k_{\mathsf{reg}} \preceq (\lambda_1 + k, \dots, \lambda_m + k, 0)$ the claim follows. $\square$

Appendix: Section 5

10.13. **Proof of Proposition 5.1.** We provide the proof in the cubic case only and thus assume $U_i = \mathbb{C}^n$. The matrix multiplication tensor $\mathrm{M}_{\underline{U}}$ is the structural tensor of the associative algebra $A = \mathrm{End}(U)$. Note that $A^\times = \mathrm{GL}(U)$. Recall that $L_a, R_b \colon A \to A$ denote the left multiplication with a and the right multiplication with b, respectively ($a, b \in A$). If we interpret $A = U \otimes U^*$, then we have $L_a R_b = a \otimes b^*$. Lemma 10.5 states that any element g of $\mathrm{stab}(\mathrm{M}_{\underline{U}})$ is of the form

$$g = \left(L^*_{\varepsilon^{-1}}(\psi^{-1})^*, R^*_{\eta^{-1}}(\psi^{-1})^*, L_\varepsilon R_\eta \psi\right)$$

for some $\varepsilon, \eta \in A^\times, \psi \in \mathrm{Aut}A$. The Skolem-Noether Theorem [16] implies that any automorphism ψ of A is of the form $\psi = L_\rho R_{\rho^{-1}}$ for some $\rho \in A^\times$. We thus obtain

$$\psi^{-1} L_{\varepsilon^{-1}} = L_{\rho^{-1}} R_\rho L_{\varepsilon^{-1}} = L_{\rho^{-1}\varepsilon^{-1}} R_\rho = \rho^{-1}\varepsilon^{-1} \otimes \rho^*,$$

which implies $L^*_{\varepsilon^{-1}}(\psi^{-1})^* = (\psi^{-1} L_{\varepsilon^{-1}})^* = ((\varepsilon\rho)^{-1})^* \otimes \rho$. Similarly, we obtain $R^*_{\eta^{-1}}(\psi^{-1})^* = (\rho^{-1})^* \otimes \eta^{-1}\rho$. Finally, $L_\varepsilon R_\eta \psi = \varepsilon\rho \otimes (\rho^{-1}\eta)^* \simeq (\rho^{-1}\eta)^* \otimes \varepsilon\rho$. (The flip $\simeq$ is due to our convention $\mathrm{Hom}(U_3, U_1) \simeq U_3^* \otimes U_1$, unlike $\mathrm{Hom}(U_2, U_1) \simeq U_1 \otimes U_2^*$, $\mathrm{Hom}(U_3, U_2) \simeq U_2 \otimes U_3^*$.) Setting $\alpha_1 = \varepsilon\rho$, $\alpha_2 = \rho$, $\alpha_3 = \eta^{-1}\rho$ we see that g has the required form. ☐

Lemma 10.8. *If the stabilizer of $w \in (U_1^* \otimes U_2) \otimes (U_2^* \otimes U_3) \otimes (U_3^* \otimes U_1)$ contains the stabilizer H of $\mathrm{M}_{\underline{U}}$, then $w = c\,\mathrm{M}_{\underline{U}}$ for some $c \in \mathbb{C}$.*

Proof. We have to show that the space of H-invariants of $(U_1^* \otimes U_2) \otimes (U_2^* \otimes U_3) \otimes (U_3^* \otimes U_1)$ is one-dimensional. Due to the description of H in Proposition 5.1 it suffices to prove that the space of $\mathrm{GL}(U_i)$-invariants of $U_i^* \otimes U_i$ is one-dimensional. The latter follows as a special case of (9.1) taking $\mu_i = (1, 0, \ldots, 0)$. ☐

10.14. **Proof of Proposition 5.2.** We follow [27, Proposition 5.2.1]. Assume that $U_i = \mathbb{C}^{n_i}$. Let $T(K_s)$ and T_s denote the maximal tori of $K_s := \mathrm{SL}(U_1) \times \mathrm{SL}(U_2) \times \mathrm{SL}(U_3)$ and G_s, respectively, consisting of triples of diagonal matrices with determinant 1. It is clear that $R_s := \Phi(T(K_s))$ is a subgroup of T_s. Since R_s is a connected subgroup of a torus, it is itself a torus and thus reductive [21].

We claim that T_s equals the centralizer of R_s in G_s. Indeed suppose that $g = (g_1, g_2, g_3) \in G_s$ commutes with all elements of R_s. Then g_1 commutes with all diagonal matrices $\mathrm{diag}(a_i b_j^{-1})$, where $a_1 \cdots a_{n_1} = 1$ and $b_1 \cdots b_{n_2} = 1$. It is possible to choose a_i, b_j such that $a_i b_j^{-1}$ are pairwise distinct. Therefore g_1 must be a diagonal matrix. Similarly, g_2, g_3 must be diagonal so that $g \in G_s$.

We apply now Theorem 10.7 to the reductive subgroup R_s of the stabilizer H of $\mathrm{M}_{\underline{U}}$. Any one-parameter subgroup $\sigma \colon \mathbb{C}^\times \to T_s$ is of the form $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t))$ with

$$\sigma_1(t) = \mathrm{diag}(t^{\mu_{ij}}),\ \sigma_2(t) = \mathrm{diag}(t^{\nu_{jk}}),\ \sigma_3(t) = \mathrm{diag}(t^{\pi_{ki}}),$$

where $\mu_{ij}, \nu_{jk}, \pi_{ki} \in \mathbb{Z}$ for $i \le n_1, j \le n_2, k \le n_3$. Since $\det\sigma_1(t) = \det\sigma_2(t) = \det\sigma_3(t) = 1$ we must have

$$\sum_{i,j} \mu_{ij} = 0,\ \sum_{j,k} \nu_{jk} = 0,\ \sum_{k,i} \pi_{ki} = 0. \tag{10.6}$$

Let $(e_{ij}), (e_{jk}), (e_{ki})$ denote the standard bases of $\mathbb{C}^{n_1\times n_2}, \mathbb{C}^{n_2\times n_3}, \mathbb{C}^{n_3\times n_1}$, respectively. The matrix multiplication tensor can then be expressed as

$$\langle n_1, n_2, n_3\rangle = \sum_{i,j,k} e_{ij} \otimes e_{jk} \otimes e_{ki}.$$

We have

$$\sigma(t)\langle n_1, n_2, n_3\rangle = \sum_{i,j,k} t^{\mu_{ij}+\nu_{jk}+\pi_{ki}} e_{ij} \otimes e_{jk} \otimes e_{ki}.$$

Suppose that the limit of $\sigma(t)\langle n_1, n_2, n_3\rangle$ for $t \to 0$ exist. Then

$$\forall i, j, k \quad \mu_{ij} + \nu_{jk} + \pi_{ki} \geq 0.$$

Summing over all i, j, k and using (10.6) we get

$$\sum_{i,j,k}(\mu_{ij} + \nu_{jk} + \pi_{ki}) = \sum_{k}\sum_{ij} \mu_{ij} + \sum_{i}\sum_{j,k} \nu_{jk} + \sum_{j}\sum_{k,i} \pi_{ki} = 0.$$

Therefore, we have $\mu_{ij} + \nu_{jk} + \pi_{ki} = 0$ for all i, j, k. We conclude that $\lim_{t\to 0} \sigma(t)\langle n_1, n_2, n_3\rangle = \langle n_1, n_2, n_3\rangle$. Theorem 10.7 implies that the G_s-orbit of $\langle n_1, n_2, n_3\rangle$ is closed. $\square$

Appendix: Section 6

10.15. **Explicit realizations of Schur-Weyl modules.** For the following well known facts see [11, 12]. Let $V = \mathbb{C}^m$ with the standard basis $e_1, \ldots, e_m$. For a partition $\lambda \vdash_m d$ we denote by $\mathcal{T}_m(\lambda)$ the set of tableaux T of shape λ with entries in $\{1, 2, \ldots, m\}$. Every $T \in \mathcal{T}_m(\lambda)$ has a content $\alpha \in \mathbb{N}^m$, where α_j counts the number of occurences of j in T.

Let St_λ denote the standard tableau arising when we number the boxes of the Young diagram of λ columnwise downwards, starting with the leftmost column, cf. Figure 2. We assign to $T \in \mathcal{T}_m(\lambda)$ the basis vector $e(T) := e_{j_1} \otimes \cdots \otimes e_{j_d} \in V^{\otimes d}$, where $j_k \in \{1, \ldots m\}$ is the entry of T at the box which is numbered k in St_λ. In other words, j_k is the kth entry of T when we read the tableau T columnwise downwards, starting with the leftmost column. Note that $e(T)$ is a weight vector with respect to the subgroup $T_m \subseteq \mathrm{GL}_m$ of diagonal matrices, and the weight of $e(T)$ equals the content α of T. One should think of the tableau T as a convenient way to record the basis vector $e(T)$.

1	4	6	8
2	5	7	
3			

FIGURE 2. St_λ for $\lambda = (4, 3, 1)$, $P_\lambda = \mathrm{Perm}\{1, 4, 6, 8\} \times \mathrm{Perm}\{2, 5, 7\} \times \mathrm{Perm}\{3\}$, $Q_\lambda = \mathrm{Perm}\{1, 2, 3\} \times \mathrm{Perm}\{4, 5\} \times \mathrm{Perm}\{6, 7\} \times \mathrm{Perm}\{8\}$.

Let P_λ the subgroup of permutations in St_d preserving the rows of St_λ and denote by Q_λ the subgroup of permutations in St_d preserving the columns of S_λ, cf. Figure 2. To any $T \in \mathcal{T}_m(\lambda)$ we assing now the following vector in $V^{\otimes d}$:

$$v(T) := \frac{1}{|P_\lambda||Q_\lambda|} \sum_{\substack{\sigma\in Q_\lambda \\ \pi\in P_\lambda}} \mathrm{sgn}(\sigma)\, \sigma\pi\, e(T).$$

Note that $v(T)$ is a weight vector and its weight equals the content α of T.

Let T_λ denote the semistandard tableau which in the ith row only has the entry i. Clearly, T_λ has the content λ. Let $\mu = \lambda'$ denote the partition dual to λ and let ℓ be the length of μ. Then we have

$$v_\lambda := v(T_\lambda) = (e_1 \wedge \ldots \wedge e_{\mu_1}) \otimes \cdots \otimes (e_1 \wedge \ldots \wedge e_{\mu_\ell}). \tag{10.7}$$

It is easy to see that v_λ is a U_m-invariant weight vector of weight λ, where $U_m \subseteq \mathrm{GL}_m$ denotes the subgroup of upper triangular matrices with ones on the main diagonal. Hence the GL_m-submodule V_λ generated by v_λ is irreducible of highest weight λ. We have

$$V_\lambda = \mathrm{span}\big\{(x_1 \wedge \ldots \wedge x_{\mu_1}) \otimes \cdots \otimes (x_1 \wedge \ldots \wedge x_{\mu_\ell}) \mid x_i \in V\big\}$$

and V_λ is spanned by $\{v(T) \mid T \in \mathcal{T}_m(\lambda)\}$. It is well known that the $v(T)$ form a basis of V_λ when T runs over all semistandard tableaux in $\mathcal{T}_m(\lambda)$. A basis of the weight space V_λ^α is provided by the $v(T)$ where $T \in \mathcal{T}_m(\lambda)$ runs over all semistandard tableaux with content α. We embed S_m into GL_m by mapping $\pi \in S_m$ to the permutation matrix P_π. Note that the group S_m acts on $\mathcal{T}_m(\lambda)$ by permutation of the entries of the tableaux. Then we have $P_\pi v(T) = v(\pi T)$ for $T \in \mathcal{T}_m(\lambda)$. We have thus found explicit realizations of Schur-Weyl modules.

Suppose now $d = m$ and consider the weight space $\mathcal{S}_\lambda := V_\lambda^{\varepsilon_m}$ of weight $\varepsilon_m = (1, \ldots, 1)$. Then $\mathcal{S}_\lambda$ is a S_m-submodule of V_λ. Moreover, the vectors $v(T)$, where T runs over all standard tableaux of shape λ, provide a basis of $\mathcal{S}_\lambda$. One can show that $\mathcal{S}_\lambda$ is irreducible and isomorphic to $[\lambda]$. We have thus also found an explicit realization of the irreducible S_m-module $[\lambda]$.

10.16. **Tableaux straightening.** An explicit description of the action the subgroup S_m of GL_m on the basis $(v(T))$ is provided by the following tableau straightening algorithm [11, p.97-99, p.110]. It takes as input any tableau $T \in \mathcal{T}_m(\lambda)$ and expresses the vector $v(T)$ as an integer linear combination of the basis vectors $v(S)$, where S is semistandard. This way, we obtain an explicit description of the operation of $\mathrm{stab}(\alpha)$ on the weight space V_λ^α, which is required for applying Theorem 4.4.

(1) If T is semistandard, return $v(T)$.
(2) If the columns of T do no have pairwise distinct entries, return 0. Otherwise, apply column permutations π to put all columns in strictly increasing order by applying the rule $\pi v(T) = \mathrm{sgn}(\pi)\, v(\pi T)$.
(3) If the resulting tableau is not semistandard, suppose the kth entry of the jth column is strictly larger than the kth entry of the $(j+1)$th column. Then we have $v(T) = \sum_S v(S)$, where S ranges over all tableaux that arise from T by exchanging the top k elements from the $(j+1)$th column with any selection of k elements in the jth column, preserving their vertical order. Continue recursively with the resulting S.

See [11, p. 110] for a proof that this algorithm terminates (whatever choice of k and j is made in step (3)).

10.17. **Explicit construction of highest weight vectors in $\mathcal{O}(W)$.** Our goal here is to give an explicit description of the space of highest weight vectors of $\mathcal{O}(W^*)_d$ that is amenable to calculations (we replaced W by W^* to simplify notation). We shall proceed in several steps.

1. Recall from (10.7) that v_λ is a highest weight vector of V_λ. We have (recall §10.1)

$$\mathcal{H}_\lambda(V^{\otimes d}) = \mathrm{span}\big\{\pi\, v_\lambda \mid \pi \in S_d\big\}. \tag{10.8}$$

This follows from Schur-Weyl duality (10.1) and the fact that $[\lambda]$ is spanned by the S_d-orbit of any of its nonzero vectors.

2. Now assume $W_i = \mathbb{C}^{m_i}$, consider $W := W_1 \otimes W_2 \otimes W_3$ of format $\underline{m}$ with the group $G = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3)$ acting on W and let $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda^+(\underline{m})$. We define $v_{\underline{\lambda}} := v_{\lambda_1} \otimes v_{\lambda_2} \otimes v_{\lambda_3} \in W^{\otimes d}$. Note that the group $(S_d)^3$ acts on $W^{\otimes d}$. Equation (10.8) implies

$$\mathcal{H}_{\underline{\lambda}}(W^{\otimes d}) = \mathcal{H}_{\lambda_1}(W_1^{\otimes d}) \otimes \mathcal{H}_{\lambda_2}(W_2^{\otimes d}) \otimes \mathcal{H}_{\lambda_3}(W_3^{\otimes d}) = \mathrm{span}\big\{\underline{\pi}\, v_{\underline{\lambda}} \mid \underline{\pi} \in (S_d)^3\big\}.$$

3. We embed S_d into $(S_d)^3$ diagonally via $\pi \mapsto (\pi, \pi, \pi)$. The induced action on $W^{\otimes d}$ just permutes the different copies of W. Consider now the projection $P_{\mathsf{Sym}} \colon W^{\otimes d} \to \mathsf{Sym}^d W$ onto the subspace of symmetric tensors given by $d!^{-1} \sum_{\pi \in S_d} \pi$. Recall also $\mathcal{O}(W^*)_d \simeq \mathsf{Sym}^d W$. We arrive at the desired characterization

$$\mathcal{H}_\lambda(\mathcal{O}(W^*)_d) = \mathcal{H}_\lambda(\mathsf{Sym}^d W) = P_{\mathsf{Sym}}\big(\mathcal{H}_\lambda(W^{\otimes d})\big) = \mathrm{span}\big\{P_{\mathsf{Sym}}(\underline{\pi}\, v_{\underline{\lambda}}) \mid \underline{\pi} \in S_d^3\big\}.$$

Let $w \in W^*$. In order to prove that $V_{\underline{\lambda}}(G)$ occurs in $\mathcal{O}(\overline{Gw})$ it is sufficient to exhibit some $\underline{\pi} \in (S_d)^3$ and some $g \in G$ such that $P_{\mathsf{Sym}}(\underline{\pi}\, v_{\underline{\lambda}})(gw) \neq 0$ as polynomial evaluation. A straightforward algorithm for this evaluation requires at least $R(w)^d$ steps and thus only allows the study of small examples in practice.

10.18. **Details of the proof of Lemma 6.1.** Recall from §4.2 that the weight space V_λ^α is invariant under the action of $\mathrm{stab}(\alpha)$. We are interested in the splitting of V_λ^α into irreducible $\mathrm{stab}(\alpha)$-modules.

Remark 10.9. In the special case $\alpha = d\varepsilon_m$, where $\mathrm{stab}(\alpha) = S_m$, it is known [13] that the arising multiplicities are special plethysm coefficients, namely

$$\mathrm{mult}([\pi], V_\lambda^{d\varepsilon_m}) = \mathrm{mult}(V_\lambda(\mathrm{GL}_m), S_\pi(\mathsf{Sym}^d \mathbb{C}^m)) \quad \text{for } \pi \vdash m.$$

Lemma 10.10. *Let $0 \le s < m$, $d \ge 1$. Then $V_{(dm-s)1^s}^{d\varepsilon_m} \simeq [(m-s)\, 1^s]$ as S_m-modules.*

Proof. Let $\mathcal{ST}$ denote the set of semistandard tableaux of shape $(dm-s)1^s$ and content $d\varepsilon_m$. Moreover, let $\mathcal{S}$ denote the set of standard tableaux of shape $(m-s)1^s$. Let $T \in \mathcal{ST}$ and suppose that $1, a_1, \ldots, a_s$ are the entries of the first column of T. After deleting $d-1$ of the boxes with the entries $1, \ldots, m$ from the first row of T, we obtain a standard tableau $\psi(T) \in \mathcal{S}$. It is clear that $\psi \colon \mathcal{ST} \to \mathcal{S}$ is a bijection. The algorithmic description of the Schur-Weyl modules above easily implies that $\psi(\pi T) = \pi \psi(T)$ for any $\pi \in S_m$. We use now that $(v(T))_{T \in \mathcal{ST}}$ and $(v(T'))_{T' \in \mathcal{S}}$ form a basis of the weight space $V_{(dm-s)1^s}^{d\varepsilon_m}$ and the S_m-module $[(m-s)\, 1^s]$ realized as a submodule of $(\mathbb{C}^m)^{\otimes m}$ as in §10.15. $\square$

Taking $d = 2$ and $s = 3$ we get $\big(V_{2\varepsilon_m}^{2\varepsilon_m} \otimes V_{2\varepsilon_m}^{2\varepsilon_m} \otimes V_{(2m-3)1^3}^{2\varepsilon_m}\big) \simeq \big([m] \otimes [m] \otimes [(m-3)\, 1^3]\big)^{S_m} = 0$.

Lemma 10.11. *As $S_{m-1} \times S_2$-modules we have*

1. $V_{(2^m)}^{2^{m-1}1^2} \simeq [m-1] \otimes [2]$.
2. $V_{(2m-3)1^3}^{2^{m-1}1^2} \simeq \big([(m-4)\, 1^3] \otimes [2]\big) \oplus \big([(m-2)\, 1] \otimes [1^2]\big) \oplus \big([(m-3)\, 1^2] \otimes [2]\big) \oplus \big([(m-3)\, 1^2] \otimes [1^2]\big)$.

Proof. 1. There is a single semistandard tableau T of shape 2^m and content $2^{m-1}\, 1^2$: the ith row contains the entries i, i for $i < m$ and the mth row contains $m, m+1$. The tableau T is

fixed by the action of S_{m-1}. The transposition π in S_2 exchanges m and $m+1$. However, the straightening algorithm shows that $v(\pi T) = v(T)$.

2. The basis of $V^{2^{m-1}1^2}_{(2m-3)1^3}$ is indexed by semistandard tableaux which fall into four different classes as indicated in Figure 3 and Figure 4.

$$\begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \cdots & m\!-\!1 & m\!-\!1 & m & m\!+\!1 \\ a \\ b \\ c \end{array} \qquad \begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \cdots & m\!-\!1 & m\!-\!1 \\ a \\ m \\ m\!+\!1 \end{array}$$

Figure 3. Tableaux of class 1 and class 2.

$$\begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \cdots & m\!-\!1 & m\!-\!1 & m\!+\!1 \\ a \\ b \\ m \end{array} \qquad \begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \cdots & m\!-\!1 & m\!-\!1 & m \\ a \\ b \\ m\!+\!1 \end{array}$$

Figure 4. Tableaux of class 3 and class 4.

Omitting the boxes with entries $m, m+1$ in the tableaux of class 1 and deleting repeated entries in the first row, we obtain a bijection of the set of tableaux of class 1 with the set of standard tableaux of shape $(m-4)\,1^3$. It follows that the span of the basis vectors of class 1 is isomorphic to $[(m-4)\,1^3] \otimes [2]$.

Similarly, the tableaux of class 2 are in bijection with the standard tableaux of the shape $(m-2)\,1^3$. The span of the basis vectors of class 2 is isomorphic to $[(m-2)\,1^3] \otimes [1^2]$ (note the sign change when permuting m with $m+1$).

Let $\mathcal{T}$ denote the set of tableaux of class 3 and consider the transposition $\pi := (m\,m+1)$. Then $\{\pi T \mid T \in \mathcal{T}\}$ is the set of tableaux of class 4. Clearly, $\mathcal{T}$ is in bijection with the standard tableaux of shape $(m-3)\,1^2$. The vectors $v(T) + \pi v(T)$ for $T \in \mathcal{T}$ span $[(m-3)\,1^2] \otimes [2]$, whereas the vectors $v(T) - \pi v(T)$ span $[(m-3)\,1^2] \otimes [1^2]$. $\square$

Lemma 10.11 implies now

$$\big(V^{2^{m-1}1^2}_{(2^m)} \otimes V^{2^{m-1}1^2}_{(2^m)} \otimes V^{2^{m-1}1^2}_{(2m-3)1^3}\big)^{S_{m-1}\times S_2} = 0,$$

which was needed in the proof of the first part of Lemma 6.1. We omit the details of the proof of the second part of this lemma, see however §10.17.

The following was claimed in Remark 6.2.

Lemma 10.12. *Let $n \geq 2$. Then $g(2^{n^2}, (2n)^n, (2n)^n) = 1$ and $g((2n^2-3)\,1^3, (2n)^n, (2n)^n) > 0$.*

Proof. The first claim follows from [8, Satz 3.1]. For the second claim, we note that $\underline{\lambda} := ((2n^2-3)\,1^3, (2n)^n, (2n)^n)$ can be decomposed as $\underline{\lambda} = \underline{\mu} + (n-1)\cdot(2n, 2^n, 2^n)$, where $\underline{\mu} := ((2n-3)\,1^3, 2^n, 2^n)$. It is clear that $g(2n, 2^n, 2^n) = 1$. It follows from [31, 33] that $g(\underline{\mu}) = g((2n-3, 1^3), (n^2), (n^2)) = 1$. Since the triples with positive Kronecker coefficients form a semigroup, the second assertion follows. $\square$

Appendix: Section 7

Proposition 10.13. *Let Z be an irreducible normal complex algebraic variety and f be a regular function defined on a nonempty Zariski open subset of Z. If f has an extension to Z which is continuous in the $\mathbb{C}$-topology, then f has a regular extension to Z.*

Proof. The proof uses some standard facts from algebraic geometry [30, III,§8]. Since Z is normal, the rational function f has a well defined divisor. Suppose that f had a pole of multiplicity $k \geq 1$ at the irreducible hypersurface H of Z. Let p denote the vanishing ideal of H. Then we can write $f = p^{-k}g/h$ for some $g, h \in \mathcal{O}(Z)$ such that $g, h \notin p$. Choose $z \in H$ such that $g(z)h(z) \neq 0$ and let z_k be any sequence in Z converging to z. Then $\lim_{k\to\infty} f(z_k) = \infty$, contradicting the assumption that f has a $\mathbb{C}$-continuous extension to Z.

Therefore, f has no pole divisor and hence f is a regular function. □

Proposition 10.14. *Let Z be an irreducible affine complex algebraic variety and $U \subseteq Z$ be a nonempty Zariski open subset, $U \neq Z$. If U is affine, then $Z \setminus U$ is of pure codimension one in Z.*

Proof. (Sketch) We use some standard facts from algebraic geometry. Let $\varphi\colon \tilde{Z} \to Z$ be the normalization of Z, cf. [15]. Then $\tilde{Z}$ and $\tilde{U} := \varphi^{-1}(U)$ are affine. Let C_i be the irreducible components of $Z \setminus U$. Since φ is finite, $C_i' := \varphi^{-1}(C_i)$ are the irreducible components of $\tilde{Z} \setminus \tilde{U}$ and $\dim C_i' = \dim C_i$. We may therefore assume that Z is normal.

Suppose first that all the irreducible components of $Z \setminus U$ have codimension at least two. Consider the injection $\iota\colon U \hookrightarrow Z$. The restriction morphism $\iota^*\colon \mathcal{O}(Z) \to \mathcal{O}(U)$ is bijective since, by a reasoning as in the proof of Proposition 10.13, any regular function on U can be extended to Z. Since Z and U are assumed to be affine, ι is an isomorphism and hence $U = Z$. We omit the proof of the general case. □

The following general result is due to [26].

Proposition 10.15. *Let G be a reductive group and H be a closed subgroup. Then G/H is affine iff H is reductive.*

10.19. **Proof of Theorem 7.1.** The following lemma settles part (1) of Theorem 7.1.

Lemma 10.16. *Suppose that $w \in W$ and $\varepsilon_{\underline{m}} \in S^o(w)$, where $\underline{m} = (\dim W_1, \dim W_2, \dim W_3)$. Then we have $m_1 = m_2 = m_3$.*

Proof. We have $\det g = 1$ for all $g \in \mathrm{stab}(w)$ by our assumption $\varepsilon_{\underline{m}} \in S^o(w)$, On the other hand, $(a\,\mathrm{id}_{m_1}, b\,\mathrm{id}_{m_2}, c\,\mathrm{id}_{m_3}) \in \mathrm{stab}(w)$ for any $a, b, c \in \mathbb{C}^\times$ with $abc = 1$. This implies

$$1 = \det(a\,\mathrm{id}_{m_1}, b\,\mathrm{id}_{m_2}, c\,\mathrm{id}_{m_3}) = a^{m_1}b^{m_2}c^{m_3} = a^{m_1-m_3}\,b^{m_2-m_3}.$$

Therefore, $m_1 = m_2 = m_3$. □

The next lemma shows part (2) of Theorem 7.1.

Lemma 10.17. *Let $w \in W \setminus \{0\}$ be stable and $u \in \overline{Gw} \setminus Gw$. Suppose that (g_n) is a sequence in G such that $\lim_{n\to\infty} g_n w = u$. Then we have $\lim_{n\to\infty} \det g_n = 0$.*

Proof. Since $G_s w$ is closed and $0 \notin G_s w$ we have

$$\varepsilon := \inf\{\|\tilde{g}w\| \mid \tilde{g} \in G_s\} = \min\{\|\tilde{g}w\| \mid \tilde{g} \in G_s\} > 0.$$

For each n there are $\tilde{g}_n \in G_s$ such that

$$g_n w = \det g_n \, \tilde{g}_n w. \tag{10.9}$$

Hence $\|g_n w\| = |\det g_n| \, \|\tilde{g}_n w\|$. Since $\lim_{n\to\infty} \|g_n w\| = \|u\|$ and $\|\tilde{g}_n w\| \geq \varepsilon > 0$ we conclude that $|\det g_n| \leq \|g_n w\|/\varepsilon$ is bounded.

If $\lim_{n\to\infty} \det g_n = 0$ were false, then there would be some nonzero limit point δ of the sequence $(\det g_n)$. After going over to a subsequence, we have $\lim_{n\to\infty} \det g_n = \delta$. From (10.9) we get $\lim_{n\to\infty} \tilde{g}_n w = \delta^{-1} u$. Hence $\delta^{-1} u \in \overline{G_s u} = G_s u$, which implies the contradiction $u \in Gw$. □

For the third part of Theorem 7.1 we note that for $w \in W$ and $g, h \in G$

$$g \det_w(hw) = \det_w(g^{-1}hw) = \det(g^{-1}h) = \det(g)^{-1}\det(h) = \det(g)^{-1}\det_w(hw). \tag{10.10}$$

If $\det_w$ had a regular extension to $\overline{Gw}$, then (10.10) shows that $\mathbb{C}\det_w$ is a submodule of $\mathcal{O}(\overline{Gw})$ of highest weight $-\varepsilon_{\underline{m}}$. Hence $\mathcal{O}(W)$ would contain an irreducible submodule of highest weight $-\varepsilon_{\underline{m}}$ as well. On the other hand, the Kronecker coefficient $g(\varepsilon_{\underline{m}})$ vanishes if $m > 1$. This contradicts (10.3) and proves that $\det_w$ is not a regular function on $\overline{Gw}$.

Proposition 10.13 combined with part (2) and part (3) implies now that $\overline{Gw}$ is not a normal variety, showing the fourth part of Theorem 7.1.

Part (5) of Theorem 7.1 follows by tracing the proof of Proposition 3.6: Let $f \in \mathcal{O}(Gw)_d$ be a highest weight vector. The restriction $\tilde{f}$ of f to $G_s w$ does not vanish since Gw is the cone generated by $G_s w$. So $\tilde{f}$ is a highest weight vector for the action of G_s. Let M denote the irreducible G_s-module generated by $\tilde{f}$. The G_s-equivariant restriction morphism $\mathrm{res}\colon \mathcal{O}(\overline{Gw}) \to \mathcal{O}(G_s w)$ is surjective since $G_s w$ is assumed to be closed. Hence there exists an irreducible G_s-module $N \subseteq \mathcal{O}(\overline{Gw})$ that maps to M under res. We have $N \subseteq \mathcal{O}(\overline{Gw})_\delta$ for some degree δ. Let F be a highest weight vector of the G_s-module N. Then $\mathrm{res}(F) = c\tilde{f}$ for some $c \in \mathbb{C}^\times$. W.l.o.g. $c = 1$.

For each $g \in G$ there exists $\tilde{g} \in G_s$ such that $gw = \det g \, \tilde{g}w$. Moreover, since F is homogeneous of degree δ,

$$F(gw) = (\det g)^\delta F(\tilde{g}w) = (\det g)^\delta f(\tilde{g}w).$$

Moreover, since f is homogeneous of degree d, $f(gw) = (\det g)^d f(\tilde{g}w)$. We conclude that

$$F(gw) = (\det g)^{\delta-d} f(gw) = (\det_w(gw))^{\delta-d} f(gw).$$

Therefore, $(\det)^{\delta-d} f = F$ is regular on $\overline{Gw}$ and the assertion follows. □

10.20. **Proof of Corollary 7.2.** (1) The first assertion is immediate from Theorem 7.1.

(2) Put $W = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $w := \langle m \rangle$. Proposition 4.1 implies that $(\det g)^2 = 1$ for all $g \in \mathrm{stab}(w)$. As in the proof of Theorem 7.1 we show that $\det_w^2\colon Gw \to \mathbb{C}$ has a continuous extension to $\overline{Gw}$.

If $\det_w^2$ had a regular extension to $\overline{Gw}$, then $\mathcal{O}(W)$ would contain an irreducible submodule of highest weight $\underline{\lambda} = (2^m, 2^m, 2^m)$ (compare (10.10)). On the other hand, using the symmetry property $g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu)$ of Kronecker coefficients [34] (with λ' denoting the transposed partition), we obtain $g(2^m, 2^m, 2^m) = g(m^2, m^2, 2^m) = 0$ for $m \geq 5$. (The vanishing since the

right hand partition has more than four rows.) This contradicts (10.3) and proves that $\det_w^2$ does not have a regular extension to $\overline{Gw}$. The assertion follows now with Proposition 10.13. □

10.21. **Proof of Proposition 7.3.** Put $H := \mathrm{stab}(w)$ and $H_s := H \cap G_s$. If w is stable, then $G_s/H_s \simeq G_s w$ is closed and hence affine. By Proposition 10.15, H_s is reductive. Consider the morphism $H \to (\mathbb{C}^\times)^3, g \mapsto (\det g_1, \det g_2, \det g_3)$ with kernel H_s. Since reductiveness is preserved under extensions, and closed subgroups of $(\mathbb{C}^\times)^3$ are reductive, it follows that H is reductive [18]. Proposition 10.15 implies that $G/H \simeq Gw$ is affine. The last assertion follows by applying Proposition 10.14 to $Z = \overline{Gw}$ and $U = Gw$. □

Appendix: Section 8

The easy proof of the following observation is left to the reader.

Lemma 10.18. *Let $\underline{\lambda}^1, \dots, \underline{\lambda}^s \in S(w)$ be a system of generators for the semigroup $S(w)$ with $\underline{\lambda}^i \in \Lambda^+_{d_i}(\underline{m})$. Then $P(w)$ is the convex hull of $\frac{1}{d_1}\underline{\lambda}^1, \dots, \frac{1}{d_s}\underline{\lambda}^s$. Moreover, any rational point of $P(w)$ is a rational convex combination of these points.*

Lemma 10.19. *If $u_{\underline{m}} \in P(w)$ then there exists $\ell \geq 1$ such that $\ell\varepsilon_{\underline{m}} \in S(w)$.*

Proof. By Lemma 10.18, $u_{\underline{m}}$ is a rational convex combination of $\frac{1}{d_1}\underline{\lambda}^1, \dots, \frac{1}{d_s}\underline{\lambda}^s$. Hence there exist integers $N_i \geq 0$, $N > 0$ such that $Nu_{\underline{m}} = \sum_i N_i \underline{\lambda}^i \in S(w)$. Moreover, $Nu_{\underline{m}} = \ell\varepsilon_{\underline{m}}$ where $\ell := N/m \in \mathbb{N}$. □

10.22. **Proof of Theorem 8.4.** For the second statement take any $\underline{\lambda} = (\lambda_{12}, \lambda_{23}, \lambda_{31})$ in $\Lambda^+_{dn}(n^2, n^2, n^2)$. Consider the rectangular partition $(d^n) = (d, \dots, d) \vdash_n dn$. The main result in [4] states that for $ij = 12, 23, 31$ there exists a positive stretching factor $k_{ij} \in \mathbb{N}$ such that $g(k_{ij}\lambda_{ij}, (k_{ij}d)^n, (k_{ij}d)^n) \neq 0$. Let k be the least common multiple of k_{12}, k_{23}, k_{31}. Then we have for $ij = 12, 23, 31$

$$g(k\lambda_{ij}, (kd)^n, (kd)^n) \neq 0.$$

Theorem 5.3 with $\mu_i = (kd)^n \vdash_n kdn$ implies that $k\underline{\lambda} \in S^o(\langle n, n, n\rangle)$. Hence $\frac{1}{dn}\underline{\lambda} \in P^o(\langle n, n, n\rangle)$. Since the set of $\frac{1}{dn}\underline{\lambda}$ is dense in $\Delta(n^2, n^2, n^2)$, we obtain $P^o(\langle n, n, n\rangle) = \Delta(n^2, n^2, n^2)$ as claimed.

The statement for the unit tensors is an easy consequence of Corollary 4.5(2). □

10.23. **Proof of Lemma 8.5.** Let $\underline{x} = \frac{1}{d}\underline{\lambda}$ with $\underline{\lambda} \in S^o(w) \cap \Lambda^+_d(\underline{m})$. Proposition 3.6 implies that there exists $k \in \mathbb{Z}$ such that $\underline{\lambda} + k\varepsilon_{\underline{m}} \in S(w)$, cf. (3.2). We may assume $k > 0$ due to Lemma 10.19 and our assumption $u_{\underline{m}} \in P(w)$. Hence the point

$$\frac{1}{d+km}(\underline{\lambda} + k\varepsilon_{\underline{m}}) = \frac{d}{d+km}\,\underline{x} + \frac{km}{d+km}\,u_{\underline{m}}.$$

lies in $P(w)$. From the convexity of $P(w)$, we conclude that $\{t\underline{x} + (1-t)u_{\underline{m}} \mid 0 \leq t \leq \delta\} \subseteq P(w)$, where $\delta := d/(d+km)$. Replacing δ by the minimum of these values for all generators of $S^o(w)$, we obtain that

$$\forall\, \underline{x} \in P^o(w)\ \forall\, 0 \leq t \leq \delta\ \ t\underline{x} + (1-t)u_{\underline{m}} \in P(w).$$

□

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